



CARIBBEAN
EXAMINATIONS
COUNCIL

Caribbean Advanced
Proficiency Examination®

SYLLABUS

PURE MATHEMATICS

CXC A6/U2/21

Effective for examinations from May–June 2023

CXC



CAPE®

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NOTE TO TEACHERS AND LEARNERS

This document CXC A6/U2/21 replaces CXC A6/U2/12 issued in 2012.

Please note that the syllabus has been revised and amendments are indicated by italics.

First issued 1999

Revised 2004

Revised 2007

Amended 2012

Revised 2021

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For access to short courses, training opportunities and teacher orientation webinars and workshops go to our Learning Institute at <https://pli.cxc.org/>.

PLEASE NOTE



This icon is used throughout the syllabus to represent key features which teachers and learners may find useful.

Introduction

The Caribbean Advanced Proficiency Examination® (**CAPE**®) is designed to provide certification of the academic, vocational and technical achievement of students in the Caribbean who, having completed a minimum of five years of secondary education, wish to further their studies. The examinations address the skills and knowledge acquired by students under a flexible and articulated system where subjects are organised in 1-Unit or 2-Unit courses with each Unit containing three Modules. Subjects examined under **CAPE**® may be studied concurrently or singly.

The Caribbean Examinations Council offers three types of certification at the **CAPE**® level. The first is the award of a certificate showing each **CAPE**® Unit completed. The second is the **CAPE**® Diploma, awarded to candidates who have satisfactorily completed at least six Units, including Caribbean Studies. The third is the **CXC**® Associate Degree, awarded for the satisfactory completion of a prescribed cluster of *ten* **CAPE**® Units including Caribbean Studies, Communication Studies *and Integrated Mathematics. Integrated Mathematics is not a requirement for the CXC® Associate Degree in Mathematics.* The complete list of Associate Degrees may be found in the **CXC**® Associate Degree Handbook.

For the **CAPE**® Diploma and the **CXC**® Associate Degree, candidates must complete the cluster of required Units within a maximum period of five years. *To be eligible for a CXC® Associate Degree, the educational institution presenting the candidates for the award, must select the Associate Degree of choice at the time of registration at the sitting (year) the candidates are expected to qualify for the award.* Candidates will not be awarded an Associate Degree for which they were not registered.

Pure Mathematics Syllabus

◆ RATIONALE

The Caribbean society is an integral part of an ever-changing world. The impact of globalisation on most societies encourages this diverse Caribbean region to revisit the education and career opportunities of our current and future citizens. A common denominator is for Caribbean societies to create among its citizens a plethora of quality leadership with the acumen required to make meaningful projections and innovations for further development. Further, learning appropriate problem-solving techniques, inherent to the study of mathematics, is vital for such leaders. Mathematics promotes intellectual development, is utilitarian and applicable to all disciplines. Additionally, its aesthetics and epistemological approaches provide solutions fit for any purpose. Therefore, Mathematics is the essential tool to empower people with the knowledge, competencies and attitudes that are required for academia as well as quality leadership for sustainability in this dynamic world.

Students doing this syllabus will have already been exposed to Mathematics in some form mainly through courses that emphasise skills in using mathematics as a tool, rather than giving insight into the underlying concepts. *This course of study is designed to utilise learner-centered approaches to teaching, learning and assessment to expand students' mathematical knowledge by exposing them to more advanced mathematical ideas, skills, and techniques, and to enhance their understanding of why and how the concepts are interconnected and can be utilised. This approach will minimise the fear associated with having to learn a multiplicity of seemingly unconnected facts, procedures and formulae. In addition, this course will show them that mathematical concepts lend themselves to generalisations, and that there is enormous scope for applications to solving real-world problems.*

This course of study incorporates the features of the Science, Technology, Engineering, and Mathematics (STEM) principles. On completion of this syllabus, students will be able to make a smooth transition to further studies in Mathematics and other related subject areas or move on to career choices where a deeper knowledge of the general concepts of Mathematics is required. It will enable students to develop and enhance twenty-first century skills including critical and creative thinking, problem solving, logical reasoning, modelling, collaboration, decision making, research, information and communications technological competencies which are integral to everyday life and for life-long learning. Students will be exposed to the underlying concepts of Mathematics to foster a deeper understanding and greater appreciation of the subject. This course provides insight into the exciting world of advanced mathematics, thereby equipping students with the tools necessary to approach any mathematical situation with confidence.

This Syllabus will contribute to the development of the Ideal Caribbean Person as articulated by the CARICOM Heads of Government. *This person will demonstrate multiple literacies, independent and critical thinking; and question the beliefs and practices of the past and present bringing this to bear on the innovative application of science and technology to problem solving. Such a person should also demonstrate a positive work ethic and values and displays creative imagination and entrepreneurship. In keeping with the UNESCO Pillars of Learning, on completion of this course of study, students will learn to know, learn to do, learn to be, learn to live together, and learn to transform themselves and society.*

◆ AIMS

The syllabus aims to:

1. provide understanding of mathematical concepts and structures, their development and the relationships between them;
2. enable the development of skills in the use of mathematical tools;
3. *make Mathematics fun, interesting, and recognizable;*
4. develop an appreciation of the idea of mathematical proof, the internal logical coherence of Mathematics, and its consequent universal applicability;
5. develop the ability to make connections between distinct concepts in Mathematics, and between mathematical ideas and those pertaining to other disciplines;
6. *develop skills such as, critical and creative thinking, problem solving, logical reasoning, modelling, collaboration, decision making, research, information communication and technological competences which are integral to everyday life and for life-long learning;*
7. *develop positive intrinsic mathematical values, such as, accuracy and rigour;*
8. develop the skills of recognising essential aspects of concrete, real-world problems, formulating these problems into relevant and solvable mathematical problems and mathematical modelling;
9. *integrate Information and Communications Technology (ICT) tools and skills in the teaching and learning processes; and,*
10. *prepare students for advanced courses in Mathematics and related areas.*

◆ SKILLS AND ABILITIES TO BE ASSESSED

The assessment will test candidates' skills and abilities in relation to three cognitive levels.

1. **Conceptual knowledge** - the ability to **recall** and understand appropriate facts, concepts and principles in a variety of contexts.
2. **Algorithmic knowledge** - the ability to **manipulate** mathematical expressions and procedures using appropriate symbols and language, logical deduction and inferences.
3. **Reasoning** - the ability to **select** appropriate strategy or select, **use** and **evaluate** mathematical models and **interpret** the results of a mathematical solution in terms of a given real-world problem and engage in problem-solving.

◆ PREREQUISITES OF THE SYLLABUS

Any person with a **good** grasp of the contents of the syllabus of the *Caribbean Secondary Education Certificate (CSEC®) course in Mathematics*, or equivalent, should be able to undertake the course. However, persons with a **good** grasp of the contents of the syllabus of the *Caribbean Secondary Education Certificate (CSEC®) course in Additional Mathematics* would be better prepared to pursue this course of study. Successful participation in the course will also depend on the possession of good verbal and written communication skills.

◆ STRUCTURE OF THE SYLLABUS

The syllabus is arranged into two (2) Units, Unit 1 which will lay the foundation, and Unit 2 which expands on, and applies, the concepts formulated in Unit 1.

It is therefore recommended that Unit 2 be taken after satisfactory completion of Unit 1 or a similar course. Completion of each Unit will be separately certified.

Each Unit consists of three Modules.

Unit 1: Algebra, Geometry and Calculus, contains three Modules each requiring approximately 50 hours. The total teaching time is therefore approximately 150 hours.

Module 1 - Basic Algebra and Functions

Module 2 - Trigonometry, *Coordinate* Geometry and Vectors

Module 3 - Calculus I

Unit 2: Complex Numbers, Analysis and Matrices, contains three Modules, each requiring approximately 50 hours. The total teaching time is therefore approximately 150 hours.

Module 1 - Complex Numbers and Calculus II

Module 2 - Sequences, Series and Approximations

Module 3 - Counting, Matrices and Differential Equations

◆ APPROACHES TO TEACHING THE SYLLABUS

The Specific Objectives indicate the scope of the content and activities that should be covered. Teachers are encouraged to utilise a learner-centered approach to teaching and learning. They are also encouraged to model the process for completing, solving, and calculating mathematical problems. It is recommended that activities to develop these skills be incorporated in every lesson using collaborative, integrative and practical teaching strategies. Note as well that additional notes and the formulae sheet are included in the syllabus.

◆ RECOMMENDED 2-UNIT OPTIONS

1. Pure Mathematics Unit 1 **AND** Pure Mathematics Unit 2.
2. Applied Mathematics Unit 1 **AND** Applied Mathematics Unit 2.
3. Pure Mathematics Unit 1 **AND** Applied Mathematics Unit 2.

◆ MATHEMATICAL MODELLING

Mathematical Modelling should be used in both Units 1 and 2 to solve real-world problems.

A. The topic Mathematical Modelling involves the following steps:

1. identify a real-world situation to which modelling is applicable;
2. carry out the modelling process for a chosen situation to which modelling is applicable; and,
3. discuss and evaluate the findings of a mathematical model in a written report.

B. The Modelling process requires:

1. a clear statement posed in a real-world situation, and identification of its essential features;
2. translation or abstraction of the problem, giving a representation of the essential features of the real-world;
3. solution of the mathematical problem (analytic, numerical, approximate);
4. testing the appropriateness and the accuracy of the solution against behaviour in the real-world; and,
5. refinement of the model as necessary.

C. Consider the two situations given below.

1. A weather forecaster needs to be able to calculate the possible effects of atmospheric pressure changes on temperature.
2. An economic adviser to the Central Bank Governor needs to be able to calculate the likely effect on the employment rate of altering the Central Bank's interest rate.

In each case, people are expected to predict something that is likely to happen in the future. Furthermore, in each instance, these persons may save lives, time, and money or change their actions in some way as a result of their predictions.

One method of predicting is to set up a mathematical model of the situation. A mathematical model is not usually a model in the sense of a scale model motor car. A mathematical model is a way of describing an underlying situation mathematically, perhaps with equations, with rules or with diagrams.

Some examples of mathematical models are:

1. Equations

(a) Business

A recording studio invests \$25 000 to produce a master CD of a singing group. It costs \$50.00 to make each copy from the master and covers the operating expenses. We can model this situation by the equation or mathematical model,

$$C = 50.00x + 25\,000$$

where C is the cost of producing x CDs. With this model, one can predict the cost of producing 60 CDs or 6 000 CDs.

Is the equation $x + 2 = 5$ a mathematical model? Justify your answer.

(b) Banking

Suppose you invest \$100.00 with a commercial bank which pays interest at 12% per annum. You may leave the interest in the account to accumulate. The equation $A = 100(1.12)^n$ can be used to model the amount of money in your account after n years.

2. Table of Values

Traffic Management

The table below shows the safe stopping distances for cars recommended by the Highway Code.

Speed m/h	Thinking Distance m	Braking Distance m	Overall Stopping Distance m
20	6	6	12
30	9	14	23
40	12	24	36
50	15	38	53
60	18	55	73
70	21	75	96

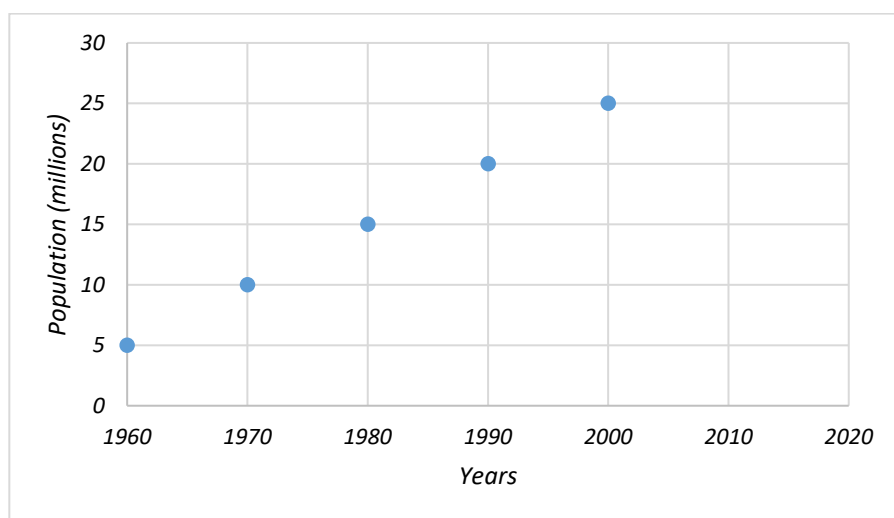
We can predict our stopping distance when travelling at 50 m/h from this model.

3. Rules of Thumb

You might have used some mathematical models of your own without realising it; perhaps you think of them as “rules of thumb”. For example, in the baking of hams, most cooks use the rule of thumb, “bake ham fat side up in roasting pan in a moderate oven (160°C) ensuring 25 to 40 minutes per $\frac{1}{2}$ kg”. The cook is able to predict how long it takes to bake *the* ham without burning it.

4. Graphs

Not all models are symbolic in nature; they may be graphical. For example, the graph below shows the population at different years for a certain country.



RESOURCE

Hartzler, J. S. and Swetz, F.

Mathematical Modelling in the Secondary School Curriculum, A Resource Guide of Classroom Exercises, Vancouver, United States of America: National Council of Teachers of Mathematics, Incorporated, Reston, 1991.

◆ UNIT 1: ALGEBRA, GEOMETRY AND CALCULUS

MODULE 1: BASIC ALGEBRA AND FUNCTIONS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. develop the ability to construct simple proofs of mathematical assertions;
2. understand the concept of a function;
3. be confident in the manipulation of algebraic expressions and the solutions of equations and inequalities;
4. understand the properties and significance of the exponential and logarithm functions; *and*,
5. develop the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

1. Reasoning and Logic

- | | | |
|-----|--|--|
| 1.1 | identify simple and compound propositions; | <i>Simple propositions, compound propositions, and connectives (disjunction, negation, conditional, bi-conditional).</i> |
| 1.2 | establish the truth value of compound statements using truth tables; | <i>Truth tables for compound statements.</i> |
| 1.3 | state the converse, contrapositive and inverse of statements; <i>and</i> , | Converse, contrapositive <i>and</i> inverse of:

(a) <i>conditional statements; and</i> ,

(b) <i>bi-conditional statements.</i> |
| 1.4 | determine whether two statements are logically equivalent. | Logical equivalence, including <i>tautology and contradiction.</i>

Identities involving propositions. |

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

2. The Real Number System – $\mathbb{R}$

2.1 use the concepts of *arithmetic operations*;

Applications of the concepts of identity, closure, inverse, commutativity, associativity, distributivity addition, multiplication and other simple binary operations.

Axioms of the system - including commutative, associative and distributive laws; non-existence of the multiplicative inverse of zero.

2.2 perform binary operations;

Binary operations.

2.3 perform *basic* operations involving surds;

Surds.

Basic operations (addition, subtraction, multiplication and rationalization) involving surds.

2.4 use the summation notation (Σ); and,

The summation of number series.

Interpretation of the summation notation.

Natural number series.

Expansion of a series.

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

2.5 *prove mathematical statements.*

Proof by mathematical induction (addition and divisibility).

Other methods of proof include direct, counter-examples.

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

3. Algebraic Operations

3.1 use the Factor Theorem;

Using Factor Theorem to:

(a) *find factors; and,*

(b) *evaluate unknown coefficients.*

3.2 use the Remainder Theorem;

Using Remainder Theorem to find the remainder.

3.3 extract all factors of $a^n - b^n$ for positive integers $n \leq 6$; and,

Factorisation.

3.4 use the order of polynomial expressions.

Order of Polynomial expressions.

4. Functions

4.1 define mathematical notations and terms related to functions;

Terms: function, ordered pairs, domain, range, one-to-one function (injective function), into function, onto function (surjective function), many-to-one, one-to-one and onto function (bijective function), composition and inverse of functions.

4.2 determine whether or not a given simple function is into, onto or one-to-one;

Functions:

(a) *injective;*

(b) *surjective;*

4.3 determine if an inverse exists for a function;

(c) *bijective; and,*

(d) *inverse.*

4.4 determine the inverse of the function f ; and,

Composite inverse.

If g is the inverse function of f , then $f[g(x)] = x$, for all x , in the domain of g .

4.5 draw graphs to show the relationship between a function and its inverse.

Transformation of a graph and its inverse.

Graphs to show relationship between the function of $y = f(x)$ given in graphical form and the inverse of $f(x)$, that is,

$$y = f^{-1}(x).$$

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

5. Exponential and Logarithmic Functions

5.1 *simplify expressions by using laws of logarithms;*

Laws of logarithms applied to problems:

(a) $\log_c (PQ) = \log_c P + \log_c Q;$

(b) $\log_c \left(\frac{P}{Q}\right) = \log_c P - \log_c Q;$ and,

(c) $\log_c P^a = a \log_c P.$

5.2 define an exponential function;

Properties of the exponential functions.

$y = a^x$ for $a \in R, x \in R$

5.3 sketch the graph of *exponential functions;*

Exponential functions and their graphs.

Graphs of the functions.

$y = a^x$ for $x > 0$

5.4 define logarithmic functions;

Properties of the logarithmic functions.

5.5 *sketch the graph of logarithmic functions;*

Natural logarithmic functions and their graphs

$y = \log_a x$

5.6 define *the* exponential function;

Properties of the exponential functions.

$y = e^x$ and its inverse $y = \ln x$, where $\ln x = \log_e x$

5.7 use the fact that $y = \ln x \Leftrightarrow x = e^y$ to *convert functions;*

Convert from logarithmic to exponential functions.

Convert from exponential to logarithmic functions.

5.8 use logarithms to solve equations of the form $a^x = b$; and,

Logarithmic solutions.

5.9 solve problems involving changing of the base of a logarithm.

Change of base.

$\log_c P = \frac{\log_a P}{\log_a c}$

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

6. The Modulus Function

6.1 define the modulus function;

Definition of the modulus function.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

6.2 use the properties of the Modulus function;

Properties:

(a) $|x|$ is the positive square root of x^2 ;

(b) $|x| < |y|$ if and only if (iff) $x^2 < y^2$; and,

(c) $|x| < |y| \Leftrightarrow \text{iff } -y < x < y$.

6.3 derive the property of the modulus function;

Properties of the modulus function.

$$|x + y| \leq |x| + |y|$$

6.4 draw graphs to show the relationship between a function and its modulus; and,

Transformation of the graphs of:

$$y = f(x) \text{ to } y = |f(x)|$$

6.5 solve equations and inequalities involving simple rational and modulus functions.

Solving equations and inequalities, using algebraic or graphical methods.

7. Cubic Functions and Equations

7.1 use the relationship among roots of an equation.

Sum of roots, the product of the roots, the sum of the product of the roots pair-wise and the coefficients of $ax^3 + bx^2 + cx + d = 0$.

Use notation of $\sum \alpha$, $\sum \alpha\beta$ and $\alpha\beta\gamma$ for

sum of roots, the sum of the product of the roots pair-wise and the product of the roots respectively.

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. *Teachers are encouraged to incorporate the use of PowerPoint Presentations or YouTube Videos to engage students in a brief review of the number systems.*
2. *Students should be encouraged to practice different methods of proof, for example, to prove that the product of two consecutive integers is an even integer and to disprove by counter-example the statement $n^2 + n - 1$ is prime $\forall n \in \mathbb{Z}$.*
3. *Students should be encouraged to work in groups to solve practice questions, using the information below. Students should explore mapping properties of quadratic functions which:*
 - (a) will, or will not, be injective, depending on which subset of the real line is chosen as the domain;
 - (b) will be surjective if its range is taken as the co-domain (completion of the square is useful here); and,
 - (c) if both injective and surjective, will have an inverse function which can be constructed by solving a quadratic equation.

Example: Use the function $f : A \rightarrow B$ given by $f(x) = 3x^2 + 6x + 5$, where the domain A is alternatively the whole of the real line, or the set $\{x \in \mathbb{R} \mid x \geq -1\}$, and the co-domain B is $\mathbb{R}$ or the set $\{y \in \mathbb{R} \mid y \geq 2\}$.

4. *Students should conduct an internet search on the theory of the quadratic equation and the nature of its roots. A class discussion should be initiated, and students asked to creatively present their findings. Teacher can also use the responses as an introduction to the lesson topic.*
5. *Guide students to apply their understanding of reasoning and logic by completing the following activity. Let p be the statement "it is cold" and let q be the statement "it is raining", Write each of the following statements in symbolic form using p and q .*
 - (a) *It is cold and it is raining.*
 - (b) *It is cold but it is not raining.*
 - (c) *It is false that it is cold, or it is raining.*
 - (d) *It is neither cold nor raining.*
 - (e) *It is not true that it is not cold or not raining.*

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

6. Students should be encouraged to apply their knowledge of logarithmic function to respond to worded problems. For example, students may be given the following statement: A Jeep Wrangler cost \$30,788 in 2005 in a certain country. 2 years later the book value on the vehicle was \$18,000. Students should be encouraged to work together to:
- (a) determine the values of a and k in the exponential model of the book value $V = ae^{kt}$; and,
 - (b) find the book value of the vehicle after 1 year and after 3 years.

RESOURCES

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Bostock, L. and Chandler, S.	<i>Core Mathematics for Advanced-Levels (4th Ed).</i> United Kingdom: Oxford University Press, 2015.
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Larson, R. and Hostetler, R.P.	<i>College Algebra (7th Ed.).</i> MA, Boston: Houghton Mifflin, 2007.

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS (cont'd)

Martin, A., Brown, K., Rigby, P. and Ridley, S. Advanced Level Mathematics Tutorials Pure Mathematics CD-ROM sample (3rd Ed.). Cheltenham, United Kingdom: Stanley Thornes (Publishers) Limited, Multi-user version and Single-user version, 2000.

Toolsie, Raymond

Pure Mathematics: A Complete Course for CAPE Unit 1. Trinidad, West Indies: Caribbean Educational Publishers Ltd, 2003.

UNIT 1

MODULE 2: TRIGONOMETRY, COORDINATE GEOMETRY AND VECTORS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. develop the ability to manipulate and describe the behaviour of trigonometric functions;
2. develop the ability to establish trigonometric identities;
3. acquire the skills to solve trigonometric equations;
4. develop the ability to represent and deal with objects in two and three dimensions through the use of coordinate geometry and vectors;
5. acquire the skills to conceptualise and to manipulate objects in two and three dimensions; and,
6. develop the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

- | | | |
|-----------|--|--|
| 1. | Trigonometric Functions, Identities and Equations (all angles will be assumed to be in radians unless otherwise stated) | |
| 1.1 | <i>use reciprocal functions of $\cos x$, $\sin x$ and $\tan x$;</i> | <i>Reciprocal functions:</i> |
| | | (a) $\sec x$; |
| 1.2 | <i>sketch graphs of reciprocal functions of $\cos x$, $\sin x$ and $\tan x$;</i> | (b) $\operatorname{cosec} x$; and, |
| | | (c) $\cot x$. |
| 1.3 | <i>derive compound-angle formulae;</i> | Compound-angle formulae for $\sin (A \pm B)$, $\cos (A \pm B)$, $\tan (A \pm B)$. |
| 1.4 | <i>use compound-angle formulae;</i> | Double-angle formulae. |
| 1.5 | <i>derive identities for the multiple-angle formulae;</i> | Multiple-angle formulae. |
| | | $\sin kA$, $\cos kA$, $\tan kA$, for $k \in \mathbb{Q}$. |

UNIT 1

MODULE 2: TRIGONOMETRY, COORDINATE GEOMETRY AND VECTORS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Trigonometric Functions, Identities and Equations (all angles will be assumed to be in radians unless otherwise stated) (cont'd)

- | | | |
|------|---|---|
| 1.6 | <i>use the factor formulae;</i> | $\sin A \pm \sin B, \cos A \pm \cos B.$ |
| 1.7 | <i>prove trigonometric identities;</i> | Trigonometric identities $\cos^2 \theta + \sin^2 \theta \equiv 1$,
$1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$, $1 + \tan^2 \theta \equiv \sec^2 \theta$. |
| 1.8 | <i>determine the general solution of trigonometric equations;</i> | General solutions of simple trigonometric equations:

$\sin (k\theta \pm \alpha) = s$;

$\cos (k\theta \pm \alpha) = s$;

$\tan (k\theta \pm \alpha) = s$; and,

where $0 < \alpha < \frac{\pi}{2}$; k is positive and
$0 \leq \theta \leq 2\pi$. |
| 1.9 | <i>express $a \cos \theta + b \sin \theta$ in the forms $r \sin (\theta \pm \alpha)$ and $r \cos (\theta \pm \alpha)$;</i> | Expression of $a \cos \theta + b \sin \theta$ in the forms $r \sin (\theta \pm \alpha)$ and $r \cos (\theta \pm \alpha)$, where r is positive, $0 < \alpha < \frac{\pi}{2}$. |
| 1.10 | <i>determine the general solution for $a \cos \theta + b \sin \theta = c$, for $a, b, c \in \mathbb{R}$;</i> | |
| 1.11 | <i>solve trigonometric equations for a given range; and,</i> | <i>Using general solutions of trigonometric equations for a given range.</i> |
| 1.12 | <i>obtain maximum or minimum values of functions of the form $a \cos \theta + b \sin \theta$ for $0 \leq \theta \leq 2\pi$.</i> | <i>Using maximum and minimum values of functions of $\sin \theta$ and $\cos \theta$.</i> |

UNIT 1

MODULE 2: TRIGONOMETRY, COORDINATE GEOMETRY AND VECTORS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

2. **Coordinate Geometry**

2.1 *derive equations of circles;*

Equation of a circle.

Centre and radius of a circle.

Graph of a circle.

2.2 *derive equations of tangents and normals to a circle;*

Properties of the circle, tangents and normals.

2.3 *determine the points of intersection of a curve with a straight line;*

Intersections between lines and curves.

2.4 *determine the points of intersection of two curves;*

2.5 obtain the Cartesian equation of a curve given its parametric representation; *and,*

Cartesian equations of curves.

Parametric representations of curves (including the parabola and ellipse).

2.6 *determine the loci of points satisfying given properties.*

Line equidistant from two given points, circle, distance of a general point from two given points.

3. **Vectors**

3.1 express a vector in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } x\mathbf{i} + y\mathbf{j} + z\mathbf{k};$$

Vectors in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

or $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of x -, y - and z -axis respectively.

3.2 *perform basic operations on vectors;*

Equality of vectors.

Addition of vectors.

Subtraction of vectors.

Multiplication of a vector by a scalar quantity.

UNIT 1

MODULE 2: TRIGONOMETRY, COORDINATE GEOMETRY AND VECTORS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Vectors (cont'd)

3.3	<i>calculate</i> the magnitude and direction of a vector;	<i>Magnitude</i> and direction of a vector.
3.4	derive unit vectors, position vectors and displacement vectors;	Position vectors, unit vectors, displacement vectors.
3.5	use unit vectors, position vectors and displacement vectors;	<i>Application of the position vectors, unit vectors, displacement vectors.</i>
3.6	<i>calculate</i> the angle between two given vectors;	Scalar (Dot) Product.
3.7	<i>determine the equation of a line with given conditions;</i>	Equation of a line <i>in the following forms:</i> (a) <i>vector;</i> (b) <i>parametric; and,</i> (c) <i>cartesian.</i> <i>Conditions for equation of a line:</i> (a) <i>a point on the line and a vector parallel to the line; or,</i> (b) <i>two points on the line.</i>
3.8	determine <i>the relationship between</i> two lines; and,	<i>Parallel lines, intersecting lines, skewed lines.</i>
3.9	<i>determine</i> the equation of a plane, in the form $ax + by + cz = d$, or $\mathbf{r} \cdot \mathbf{n} = d$.	Equation of a plane: (a) <i>given a point in the plane and the normal to the plane; and,</i> (b) <i>given 3 points in the plane and the normal to the plane (cross product will not be required).</i>

UNIT 1

MODULE 2: TRIGONOMETRY, COORDINATE GEOMETRY AND VECTORS (cont'd)

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. Teachers are encouraged to engage students in a guided class discussion after demonstrating the process to master proofs of Trigonometric Identities using identities such as the formulae for: $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$, $\sin 2A$, $\cos 2A$, $\tan 2A$. Reinforce by showing that the identity $\frac{1 - \cos 4\theta}{\sin 4\theta} \equiv \tan 2\theta$ can be established by deducing that $\cos 4\theta \equiv 1 - 2\sin^2 2\theta$ and $\sin 4\theta \equiv 2\sin 2\theta \cos 2\theta$.
2. Students should be engaged in activities requiring them to derive trigonometric functions $\sin x$ and $\cos x$ for angles x of any value (including negative values), using the coordinates of points on the unit circle.
3. Teachers are encouraged to incorporate the use of PowerPoint Presentations or Open Educational Resources such as YouTube videos to introduce students to the three-dimensional axis and help them understand how to plot vectors in three dimensions. Students should then be engaged in practice activities.
4. Teachers are encouraged to incorporate the use of explainer videos to reinforce the concept of Coordinate Geometry. Students should then be encouraged to work in groups to solve worded problems and provide explanations on how they arrived at the answers. For example:

The coordinates of the point L, M, and N are (8,-2), (-8, 10) and (14,6) respectively.

- (a) Use your knowledge of gradients to show that the three points form a right-angle triangle, and state which point has the right angle.
- (b) A circle passes through the points L, M, and N. Determine the coordinates of O, the centre of the circle.
- (c) A point P lies on the circumference of the circle:
 - (i) determine the coordinates of P if $OP = LN$; and,
 - (ii) describe the shape of OLNP.

UNIT 1

MODULE 2: TRIGONOMETRY, COORDINATE GEOMETRY AND VECTORS (cont'd)

RESOURCES

<i>Bahall, D.</i>	<i>Pure Mathematics Unit 1 for CAPE Examinations. Macmillan Publishers Limited, 2013.</i>
Bostock, L. and Chandler, S.	<i>Core Mathematics for Advanced-Levels (4th Ed.).</i> United Kingdom: <i>Oxford University Press</i> , 2015.
Campbell, E.	Pure Mathematics for CAPE, Vol. 1. Jamaica: LMH Publishing Limited, 2007.
<i>Toolsie, Raymond</i>	<i>Pure Mathematics: A Complete Course for CAPE Unit 1. Trinidad, West Indies: Caribbean Educational Publishers Ltd, 2003.</i>

Website:

https://www.mathworks.com/products/matlab.html?s_tid=hp_products_matlab

UNIT 1

MODULE 3: CALCULUS I

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *understand the concept of continuity of a function from its graph;*
2. *develop the ability to identify and determine the limits (when they exist) of functions in simple cases;*
3. *know the relationships between the derivative of a function at a point and the behaviour of the function and its tangent at that point;*
4. *know the relationship between integration and differentiation;*
5. *know the relationship between integration and areas under the curve, and volumes of revolution; and,*
6. *develop the ability to use concepts to model and solve real-world problems.*

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

1. Limits

- 1.1 use graphs to determine *the limit of a function*;

Concept of limit of a function.

- 1.2 describe the behaviour of a function $f(x)$;

Graphs of functions:

(a) *with and without limits; and,*

(b) *as x gets arbitrarily close to some given fixed number, using a descriptive approach.*

- 1.3 use the limit notation;

Limit Theorems:

- 1.4 use limit theorems in simple problems;

$$\lim_{x \rightarrow a} f(x) = L, f(x) \rightarrow L \text{ as } x \rightarrow a;$$

$$\text{if } \lim_{x \rightarrow a} f(x) = F, \lim_{x \rightarrow a} g(x) = G; \text{ and,}$$

$$\text{then } \lim_{x \rightarrow a} kf(x) = kF, \lim_{x \rightarrow a} f(x)g(x) = FG.$$

UNIT 1
MODULE 3: CALCULUS I (cont'd)

SPECIFIC OBJECTIVES

Students should be able to:

Limits (cont'd)

- 1.5 use the fact that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$; and,
- 1.6 determine points of continuity and discontinuity of functions.

2. Differentiation I

- 2.1 define the derivative of a function at a point as a limit;
- 2.2 use differentiation notations;
- 2.3 differentiate simple functions from first principles;

CONTENT

where k is a constant

$$\lim_{x \rightarrow a} \{f(x) + g(x)\} = F + G$$

and, provided $G \neq 0$, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{F}{G}$.

Geometric approach to limits.

(L'Hopital's rule will not be tested).

Continuity and Discontinuity.

Left-hand or right-hand limits.

Using graphs.

Using algebraic functions.

The Derivative as a limit.

First derivatives:

$$\frac{dy}{dx} \text{ OR } f'x.$$

Gradient function, differentiation from first principles.

Functions such as:

- (a) $f(x) = k$ where $k \in \mathbb{R}$;
- (b) $f(x) = x^n$, where
 $n \in \{-3, -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2, 3\}$;
- (c) $f(x) = \sin x$; and,
- (d) $f(x) = \cos x$.

UNIT 1
MODULE 3: CALCULUS I (cont'd)

SPECIFIC OBJECTIVES

Students should be able to:

Differentiation I (cont'd)

- 2.4 use the *sum*, product and quotient rules of differentiation;
- 2.5 *differentiate sums, products and quotients of trigonometric functions;*
- 2.6 apply the chain rule in differentiation;
- 2.7 solve problems involving rates of change;
- 2.8 use the sign of the *first* derivative to investigate *when* a function is increasing or decreasing;
- 2.9 apply the concept of stationary points;
- 2.10 calculate second derivatives;
- 2.11 interpret the significance of the sign of the second derivative;
- 2.12 use the sign of the second derivative to determine the nature of stationary points;
- 2.13 sketch graphs of *given functions*;

CONTENT

Differentiation of *polynomial functions*.

Trigonometric functions.

Chain rule of differentiation of:

- (a) *composite functions (substitution); and,*
- (b) *functions given by parametric equations.*

Rates of change.

Increasing and decreasing functions (simple polynomial functions).

Stationary (critical) points (maximum, minimum and point of inflection).

Second derivatives of functions (simple polynomial functions, sine and cosine functions).

Second derivative sign test.

Curve sketching, including horizontal and vertical asymptotes:

- (a) *polynomials;*
- (b) *rational functions; and,*
- (c) *trigonometric functions.*

The features of the function and its first and second derivatives.

UNIT 1
MODULE 3: CALCULUS I (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Differentiation I (cont'd)

2.14 describe the behaviour of graphs for large values of the independent variable; and,

Graphs of polynomials, rational functions and trigonometric functions.

Behaviours of functions at their end points.

2.15 use differentiation to determine equations of tangents and normals to curves.

Application of differentiation to tangents and normals to curves.

3. Integration I

3.1 show integration as the reverse process of differentiation;

Integration as the reverse of differentiation.

3.2 use the integration notation $\int f(x)dx$;

3.3 determine the indefinite integral;

The indefinite integral including the constant of integration.

Indefinite integral as a family of functions which differ by a constant.

3.4 calculate the constant of integration given certain conditions;

Conditions of integration.

The constant of integration.

3.5 demonstrate the use of integration theorems;

Linearity of integration.

(a) $\int cf(x) dx = c \int f(x) dx$, where c is a constant; and,

(b) $\int \{f(x) \pm g(x)\} dx =$

$$\int f(x) dx \pm \int g(x) dx$$

UNIT 1
MODULE 3: CALCULUS I (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Integration I (cont'd)

3.6 determine *indefinite integrals using integration theorems*;

Polynomial *functions*.

Simple trigonometric functions:

$$\sin(ax + b), \cos(ax + b)$$

3.7 integrate using *given substitution*;

Integration *using the method of* substitution.

3.8 use the results of *definite integrals*;

Definite integrals.

$$(a) \quad \int_a^b f(x) \, dx = \int_a^b f(t) \, dt;$$

$$(b) \quad \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \text{ for } a > 0; \text{ and,}$$

$$(c) \quad \int_a^b f(x) \, dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

3.9 apply integration to *areas under the curve*;

Applications of integration *to areas*:

(a) *bounded by the curve and one or more lines; and,*

(b) *between two curves.*

3.10 *calculate volumes of revolutions*;

Applications of integration to volumes of revolution.

Rotating regions about the x-axis or y-axis and a line.

UNIT 1
MODULE 3: CALCULUS I (cont'd)

SPECIFIC OBJECTIVES

CONTENTS

Students should be able to:

Integration I (cont'd)

- | | |
|--|---|
| <p>3.11 <i>formulate a differential equation of the form $y' = f(x)$;</i></p> <p>3.12 <i>solve differential equations of the form $y' = f(x)$; and,</i></p> <p>3.13 <i>interpret solutions from differential equations of the form $y' = f(x)$.</i></p> | <p><i>Simple first order differential equations of the type $y' = f(x)$, where $f(x)$ is a polynomial or a trigonometric function, with or without initial boundary conditions.</i></p> |
|--|---|

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. *Students should be encouraged to work in small groups. Each group should be given the following worded problem. Teachers are also encouraged to utilise similar problems to reinforce concepts.*

A certain chemical substance dissolves in water at a rate proportional to the product of the amount (in grams) undissolved and $(1/2 - D)$, where D is the ratio of the amount (in grams) dissolved to the amount (in grams) of water. When 30 grams of the substance are agitated initially with 100 grams of water, it is discovered that 10 grams of the substance are dissolved after 2 hours.

Students should be encouraged to discuss the problem and respond to the questions below. Once completed each group can be asked to explain how they arrive at their answer:

- (a) *develop a differential equation showing this information;*
- (b) *solve, completely, this differential equation; and,*
- (c) *calculate the approximate amount of grams of the chemical that would have been dissolved after 5 hours.*

UNIT 1

MODULE 3: CALCULUS I (cont'd)

2. *Incorporate the use of real-world problems that require students to analyse data and propose solutions.*
3. *The following statement can be projected or written for students to read, 'A hemispherical bowl of radius r centimetres is being filled with molten lead at a constant rate'. Students should then be encouraged to:*
 - (a) *obtain an expression for the volume of lead in the bowl when the depth of the lead in the bowl is h centimetres;*
 - (b) *show that between the time when the molten lead is halfway to the top of the bowl, and the time when the bowl is about to overflow, the rate at which the depth is rising has fallen by a quarter ($\frac{1}{4}$).; and,*
 - (c) *recommend why the rate will be reducing.*

Selected students or groups can be asked to respond to the questions and explain how they arrived at their answer.

4. *Students should be encouraged to work in groups. Each group should be given a topic to guide their research, for example one group can be asked to look at differentiation and another integration. Groups should be encouraged to gather data on definitions, formulas, and the importance of each area of study. Students should also creatively present their findings to the class.*

RESOURCES

Bahall, D.	<i>Pure Mathematics Unit 1 for CAPE Examinations. Macmillan Publishers Limited, 2013.</i>
Bostock, L., and Chandler, S.	<i>Core Mathematics for Advanced-Levels (4th Ed). United Kingdom: Oxford University Press, 2015.</i>
Campbell, E.	<i>Pure Mathematics for CAPE (Vol. 1). Jamaica: LMH Publishing Limited, 2007.</i>
Caribbean Examinations Council	<i>Area under the Graph of a Continuous Function. Barbados: Caribbean Examinations Council, 1998.</i>
Toolsie, Raymond	<i>Pure Mathematics: A Complete Course for CAPE Unit 1. Trinidad West Indies: Caribbean Educational Publishers Ltd, 2003.</i>

◆ UNIT 2: COMPLEX NUMBERS, ANALYSIS AND MATRICES

MODULE 1: COMPLEX NUMBERS AND CALCULUS II

GENERAL OBJECTIVES

On completion of this Module, students should:

1. understand *how to write* complex roots of quadratic equations;
2. develop the ability to represent objects geometrically through the use of complex numbers;
3. be confident in using the techniques of differentiation and integration; and,
4. develop the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

1. Complex Numbers

- 1.1 express complex numbers in the form $a + bi$ where a and b are real numbers;

Real and imaginary parts of a complex number.

Rectangular form of a complex number.

- 1.2 *perform the arithmetic processes on complex numbers in the form $a + bi$, where a and b are real numbers;*

Addition of complex numbers in the form $a + bi$ where a and b are the real and imaginary parts, respectively, of the complex number.

Subtraction of complex numbers in the form $a + bi$ where a and b are the real and imaginary parts, respectively, of the complex number.

Multiplication of complex numbers in the form $a + bi$ where a and b are the real and imaginary parts, respectively, of the complex number.

Division of complex numbers in the form $a + bi$ where a and b are the real and imaginary parts, respectively, of the complex number.

- 1.3 *compute the roots of the general quadratic equation using complex numbers;*

Nature of roots of a quadratic equation, imaginary roots, sums and products of roots.

$$ax^2 + bx + c = 0, \text{ when } b^2 - 4ac < 0$$

UNIT 2

MODULE 1: COMPLEX NUMBERS AND CALCULUS II (cont'd)

SPECIFIC OBJECTIVES

Students should be able to:

Complex Numbers (cont'd)

CONTENT

- | | | |
|------|---|---|
| 1.4 | use the concept that complex roots of equations with constant coefficients occur in conjugate pairs; | Conjugate pairs of complex roots. |
| 1.5 | calculate the square root of a complex number; | <i>Square root of a complex number.</i> |
| 1.6 | <i>calculate</i> the modulus of a given complex number; | The modulus of a complex number. |
| 1.7 | <i>calculate</i> the principal value of the argument θ of a non-zero complex number where $-\pi < \theta \leq \pi$; | <i>The argument of a complex number.</i> |
| 1.8 | represent complex numbers on an Argand diagram; | Representation of complex numbers (<i>their sums, differences, and product</i>) on an Argand diagram. |
| 1.9 | interpret modulus and argument of complex numbers on an Argand diagram; | |
| 1.10 | <i>determine</i> the locus of z on the Argand diagram; | Locus of a point.

The set of all points z (locus of z) on the Argand diagram such that z satisfies given properties.

<i>Description of the locus z satisfying the equations:</i>

(a) $ z - a = k$;

(b) $ z - a = z - b $; and,

(c) $\arg(z - a) = \alpha$

<i>where a and b are complex numbers, k is a constant and $-\pi < \alpha < \pi$.</i> |
| 1.11 | apply De Moivre's theorem for integral values of n ; and, | De Moivre's theorem for integral values of n . |
| 1.12 | use $e^{ix} = \cos x + i \sin x$, for real x . | Polar-argument and exponential forms of complex numbers. |

UNIT 2

MODULE 1: COMPLEX NUMBERS AND CALCULUS II (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

2. Differentiation II

- 2.1 determine the derivative of $e^{f(x)}$;
- 2.2 determine the derivative of $\ln f(x)$;
- 2.3 apply the chain rule to *differentiation of parametric equations*;
- 2.4 use the concept of implicit differentiation;
- 2.5 differentiate inverse trigonometric functions;
- 2.6 differentiate any combinations of *functions*;
- 2.7 determine second derivatives, $f''(x)$; and,
- 2.8 determine the first and second partial derivatives of $u = f(x, y)$.

Application of the chain rule to differentiation of exponential and logarithmic functions where $f(x)$ is a differentiable function of x (polynomial or trigonometric).

First derivative of a function which is defined parametrically.

Gradients of tangents and normal of *parametric equations*.

Implicit differentiation *with the assumption that one of the variables is a function of the other*.

Differentiation of inverse trigonometric functions ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$).

Differentiation of combinations of functions- polynomials, trigonometric, exponential and logarithmic.

Second derivative (*that is, $f''(x)$ OR $\frac{d^2y}{dx^2}$*) of the functions in 2.4, 2.5, 2.6).

Partial derivatives and notations.

3. Integration II

- 3.1 decompose a rational function into a sum of partial fractions;
- 3.2 integrate rational functions;

Rational functions (proper and improper) whose denominators are:

- (a) *distinct linear factors;*
- (b) *repeated linear factors;*
- (c) *distinct quadratic factors;*
- (d) *repeated quadratic factors; and,*

UNIT 2

MODULE 1: COMPLEX NUMBERS AND CALCULUS II (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Integration II (cont'd)

		(e) combinations of (a) to (d) above (repeated factors will not exceed power 2).
3.3	integrate trigonometric functions using appropriate trigonometric identities;	Trigonometric identities.
3.4	integrate exponential functions and logarithmic functions;	Exponential and logarithmic functions.
3.5	integrate functions of the form $\frac{f'(x)}{f(x)}$;	$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$
3.6	use appropriate substitutions to integrate functions;	Integration by substitution. (the substitution may be given)
3.7	use integration by parts for combinations of functions;	Integration by parts.
3.8	integrate inverse trigonometric functions;	Inverse trigonometric functions.
3.9	use integration to derive reduction formulae;	Reduction formulae.
3.10	use reduction formulae to obtain integrals of higher power; and,	Integration using reduction formula.
3.11	use the trapezium rule as a method for approximating the area under the graph of a function.	Area under the graph of a continuous function (Trapezium Rule).

UNIT 2

MODULE 1: COMPLEX NUMBERS AND CALCULUS II (cont'd)

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

1. *Encourage students to calculate the principal argument by either solving:*

- (a) *the simultaneous equations*

$$\cos \theta = \frac{\operatorname{Re}(z)}{|z|} \text{ and } \sin \theta = \frac{\operatorname{Im}(z)}{|z|}, \text{ with } -\pi < \theta \leq \pi;$$

or,

- (b) *the equation*

$$\tan \theta = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \text{ for } \operatorname{Re}(z) \neq 0 \text{ and } -\pi < \theta \leq \pi;$$

together with the representation of z on the Argand diagram.

2. *Allow students to find the loci of z -satisfying equations such as:*

- (a) $|z - a| = k;$

- (b) $|z - c| = |z - b|;$ and,

- (c) $\arg(z - a) = \alpha .$

3. *Use $e^{ix} = \cos x + i \sin x$ in integration of $\int e^{ax} \cos bx \, dx$ and $\int e^{ax} \sin bx \, dx$.*

4. *Encourage students to work in groups. Each group should be encouraged to explain how De Moivre's Theorem can be used to evaluate $(-3 + 3i)^{12}$. Selected groups should be encouraged to present the results of their discussion.*

RESOURCES

Bahall, D.	<i>Pure Mathematics Unit 2 for CAPE Examinations. Macmillan Publishers Limited, 2013.</i>
Bostock, L. and Chandler, S.	<i>Core Mathematics for Advanced-Levels (4th Ed). United Kingdom: Oxford University Press, 2015.</i>
Bradie, B.	<i>Rate of Change of Exponential Functions: A Precalculus Perspective, Mathematics Teacher Vol. 91(3), p. 224 – 237.</i>

UNIT 2

MODULE 1: COMPLEX NUMBERS AND CALCULUS II (cont'd)

- Campbell, E. *Pure Mathematics for CAPE, Vol. 2. Jamaica: LMH Publishing Limited, 2007.*
- Caribbean Examinations Council *The Exponential and Logarithmic Functions – An Investigation. Barbados: Caribbean Examinations Council, 1998.*
- Martin, A., Brown, K., Rigby, P. and Ridley, S. *Pure Mathematics. Cheltenham, United Kingdom: Stanley Thornes (Publishers) Limited, 2000.*
- Toolsie, Raymond *Pure Mathematics: A Complete Course for CAPE Unit 2. Trinidad West Indies: Caribbean Educational Publishers Ltd, 2003.*

UNIT 2

MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *understand* the concept of a sequence as a function from the natural numbers to the real numbers;
2. *understand* the difference between sequences and series;
3. distinguish between convergence and/or divergence of some standard series or sequences;
4. apply successive approximations to roots of equations; and,
5. *develop* the ability to use concept to model and solve real-world problems.

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

1. Sequences

- | | | |
|-----|--|---|
| 1.1 | <i>define a sequence $\{a_n\}$;</i> | Definition of a sequence in terms of a_n where n is a positive integer. |
| 1.2 | <i>use the formula for the n^{th} term to write a specific term in the sequence;</i> | Sequences defined by recurrence relations. |
| 1.3 | <i>describe the behaviour of convergent and divergent sequences;</i> | Convergent and divergent sequences.

Limit of a sequence.
(test of divergence/convergence is not required) |
| 1.4 | <i>identify periodic and oscillating sequences; and,</i> | Periodic and oscillating sequences. |
| 1.5 | <i>apply mathematical induction to establish properties of sequences.</i> | Using the process of mathematical induction. |

UNIT 2

MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

2. Series

2.1 *distinguish between series and sequences;*

Definition of a series, as the sum of the terms of a sequence.

Differences between series and sequences.

2.2 *write the general term of a series, using the summation notation;*

Arithmetic and Geometric series.

Making distinctions between arithmetic and geometric series.

2.3 *calculate the m^{th} partial sum S_m as the sum of the first m terms of a given series;*

Series.

Use of summation notation $\sum$.

$$S_m = \sum_{r=1}^m a_r;$$

2.4 *apply mathematical induction to establish properties of sequences and series;*

Further applications of mathematical induction to sequences and series.

2.5 *find the sum to infinity of a convergent series;*

Convergence and/or divergence of series to which the method of differences can be applied.

2.6 *apply the method of differences to appropriate series, and find their sums;*

2.7 *use the Maclaurin theorem for the expansion of series; and,*

The Maclaurin series.

2.8 *use the Taylor theorem for the expansion of series.*

The Taylor series.

UNIT 2

MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS (cont'd)

SPECIFIC OBJECTIVES	CONTENT
Students should be able to:	
3. The Binomial Theorem	
3.1 use properties of $n!$ and $\binom{n}{r}$ or ${}^nC_r$, where $n, r \in \mathbb{Z}^+$;	Factorials and Binomial coefficients; their interpretation and properties.
3.2 expand $(a + b)^n$ for $n \in \mathbb{Q}$; and,	The Binomial Theorem.
3.3 apply the Binomial Theorem to real world problems.	Applications of the Binomial Theorem.
4. Roots of Equations	
4.1 use intermediate value theorem to test for the existence of a root of $f(x) = 0$ where f is continuous;	Intermediate Value Theorem.
4.2 use interval bisection to find an approximation for a root in a given interval;	Interval Bisection.
4.3 use linear interpolation to find an approximation for a root in a given interval;	Linear interpolation.
4.4 explain, in geometrical terms, the working of the Newton-Raphson method;	Newton - Raphson Method (<i>including failure cases</i>).
4.5 use the Newton-Raphson method to compute successive approximations to the roots of $f(x) = 0$, where f is differentiable; and,	
4.6 use a given iteration to determine a root of an equation to a specified degree of accuracy.	<i>Using the method of Iteration.</i>

UNIT 2

MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS (cont'd)

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the learning activities listed below.

1. Teachers are encouraged to use instructional videos to introduce Binomial Theorem. Appropriate examples should be utilised to guide students through the process.
2. Teachers are encouraged to use examples to concretize students understanding of the Value Theorem. The following scenario can then be projected and used to guide a discussion, 'A taxi is travelling at 5 km/h at 8:00 a.m. Fifteen minutes later the speed is 100 km/h. Since the speed varies continuously, clearly at some time between 8:00 a.m. and 8:15 a.m. the taxi was travelling at 75 km/h.' Note that the taxi could have travelled at 75 km/h at more than one time between 8:00 a.m. and 8:15 a.m.
3. Students should be encouraged to illustrate the Intermediate Value Theorem using examples of continuous functions using the example: $f(x) = x^2 - x - 6$ examined on the intervals $(-2.5, -1.5)$ and $(2.5, 3.5)$.
4. Guide students in determining an interval in which a real root lies. If $f(a)$ and $f(b)$ are of opposite signs, and f is continuous, then $a < x < b$, for the equation $f(x) = 0$.
5. Students should be encouraged to work in small groups to investigate $x = \frac{a+b}{2}$ and note the resulting sign to determine which side of $\frac{a+b}{2}$ the root lies. Students should be further encouraged to repeat the method until the same answer to the desired degree of accuracy is obtained.
6. Teachers are encouraged to engage students in activities on linear interpolation.
7. Teachers are encouraged incorporate the use of real-world problems that require students to analyse data and propose solutions.

RESOURCES

Bahall, D.	Pure Mathematics Unit 2 for CAPE Examinations. Macmillan Publishers Limited, 2013.
Bostock, L. and Chandler, S.	Core Mathematics for Advanced-Levels (4 th Ed). United Kingdom: Oxford University Press, 2015.
Campbell, E.	Pure Mathematics for CAPE, Vol. 2. Jamaica: LMH Publishing Limited, 2007.
Toolsie, Raymond	Pure Mathematics: A Complete Course for CAPE Unit 2. Trinidad West Indies: Caribbean Educational Publishers Ltd, 2003.

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS

GENERAL OBJECTIVES

On completion of this Module, students should:

1. *develop* the ability to analyse and solve simple problems dealing with choices and arrangements;
2. *develop an understanding of* the algebra of matrices;
3. *develop* the ability to analyse and solve systems of linear equations;
4. *develop skills to model some real-world phenomena by means of differential equations, and solve these; and,*
5. *develop* the ability to use concepts to model and solve real-world problems.

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

1. Counting

- 1.1 *explain* the principles of counting;

Factorial notation.

Fundamental principles of counting:

- (a) *addition;*
- (b) *multiplication;*
- (c) *permutations; and,*
- (d) *combinations.*

- 1.2 *determine the number of ways of arranging n objects;*

Permutations with and without repetitions.

Arranging objects:

- (a) *in a line or a circle;*
- (b) *with restrictions;*
- (c) *without restrictions; and,*
- (d) *some of which are identical.*

- 1.3 *determine the number of ways of choosing r distinct objects from a set of n distinct objects;*

Combinations, with or without restrictions.

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Counting (cont'd)

- 1.4 use diagrams to *illustrate the sample space*;

Construction of:

- (a) *possibility space diagram;*
- (b) *venn diagram (no more than three sets); and,*
- (c) *tree diagram (no more than three branches at the first level and no more than three levels).*

- 1.5 identify the number of possible outcomes in a given sample space;

Space diagrams.

- 1.6 *define $P(A)$;*

Concept of probability.

- 1.7 *use the fact that $0 \leq P(A) \leq 1$;*

Basic probability rules and elementary applications.

- 1.8 *use the property $\sum P(x) = 1$ for all;*

- 1.9 *use the property that $P(A') = 1 - P(A)$, where A' is the event A does not occur;*

- 1.10 *use the property $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for events A and B ;*

- 1.11 *use the property $P(A \cap B) = 0$ or $P(A \cup B) = P(A) + P(B)$, where A and B are mutually exclusive events;*

- 1.12 *use the property $P(A \cap B) = P(A) \times P(B)$, where A and B are independent events; and,*

- 1.13 *use the property*

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

where $P(B) \neq 0$.

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

2. Matrices and Systems of Linear Equations

- 2.1 *determine properties of $m \times n$ matrices, for $1 \leq m \leq 3$, and $1 \leq n \leq 3$;*

Square matrix.

Zero matrix.

Identity matrix.

Singular matrix.

Non Singular matrix.

Equal matrices.

Associativity.

Commutativity.

Distributivity.

- 2.2 *perform simple operations with conformable matrices;*

Addition and subtraction.

Transposing.

Multiplication of a matrix by a scalar.

Multiplication of a matrix by a matrix.

- 2.3 *evaluate determinants;*

Determinants of 2×2 or 3×3 matrices.

Cofactors.

- 2.4 *invert $n \times n$ matrices;*

Square matrices.

Singular and Non-singular matrices.

Cofactors.

Adjoint.

Unit matrix.

Multiplicative inverse.

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Matrices and Systems of Linear Equations (cont'd)

- | | | |
|-----|---|---|
| 2.5 | <i>express a system of linear equations in matrix form;</i> | <i>3 x 3 systems of linear equations.</i> |
| 2.6 | <i>use the method of row-reduction to bring an augmented matrix with 2 or 3 unknowns to echelon form;</i> | <i>Consistency of the systems (number of solutions).
Solution by row-reduction to echelon form.</i> |
| 2.7 | <i>determine whether the system (matrix) is consistent;</i> | <i>Equivalence of the systems.</i> |
| 2.8 | <i>evaluate all solutions of a consistent system; and,</i> | <i>Solution by reduction to echelon form, $n = 2, 3$.</i> |
| 2.9 | <i>solve a system of linear equations.</i> | <i>Inverting a matrix.</i> |

Multiplication of matrices.

Row reduction of an augmented matrix, $n = 2, n = 3$.

Cramer's rule.

3. Differential Equations and Modelling

- | | | |
|-----|--|---|
| 3.1 | <i>solve first order linear differential equations; and,</i> | <i>Formulation and solution of differential equations of the form $y' - ky = f(x)$, where k is a real constant or a function of x, and f is a function.</i> |
|-----|--|---|

Particular solutions of first order linear differential equations.

Using integrating factor and boundary conditions.

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)

SPECIFIC OBJECTIVES

CONTENT

Students should be able to:

Differential Equations and Modelling (cont'd)

3.2 solve second order ordinary differential equations.

Second order ordinary differential equations with constant coefficients of the form:

1. $ay'' + by' + cy = 0$
where $a, b, c \in \mathbb{R}$

The auxiliary equation may consist of:

- (a) 2 real and distinct roots;
- (b) 2 equal roots; or,
- (c) 2 complex roots.

2. $ay'' + by' + cy = f(x)$
where $f(x)$ is:

- (a) a polynomial;
- (b) an exponential function; or,
- (c) a trigonometric function.

Solution consists of a complementary function (CF) and a particular integral (PI).

- 3. *Using boundary conditions.*
- 4. *Using given substitution to reduce to a suitable form.*

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)

Suggested Teaching and Learning Activities

To facilitate students' attainment of the objectives of this Module, teachers are advised to engage students in the teaching and learning activities listed below.

Systems of Linear Equations in Two Unknowns

1. *Students should be encouraged to plot on graph paper the pair of straight lines represented by a given pair of linear equations in two unknowns, and to examine the relationship between the pair of straight lines in the cases where the system of equations has been shown to have:*
 - (a) one solution;
 - (b) many solutions; and,
 - (c) no solutions.
2. *Teachers are encouraged to utilise explainer videos and/or PowerPoint presentations to demonstrate to students that given a system of equations with a unique solution, there exist equivalent systems, obtained by row-reduction, having the same solution. Teachers should then ask students to plot on the same piece of graph paper all the straight lines represented by the successive pairs of linear equations which result from each of the row operations used to obtain the solution.*
3. *Students should be encouraged to work in groups to respond to similar scenarios as seen in the following: 'A police department uses a computer imaging app to create digital photographs of alleged suspects of a crime from eyewitness accounts. One software package contains 180 hairlines, 90 sets of eyes and eyebrows, 75 noses, 100 mouths, and 70 chins and cheek structures.' Students should then be encouraged to provide responses to the following questions with explanations of how they arrived at their answers or solutions.*
 - (a) *How many different faces can the software package create?*
 - (b) *The suspect was wearing a face mask and as a result the eyewitness can only describe the hairline, eyes, and eyebrows of the suspect. How many different faces can be produced with this information?*
 - (c) *Calculate the probability that the digital photograph produced is the actual photograph of the suspect.*
4. *Teachers are encouraged to incorporate the use of real-world problems that require students to analyse data and propose solutions.*

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS (cont'd)

RESOURCES

<i>Bahall, Dipchand.</i>	<i>Pure Mathematics Unit 2 for CAPE Examinations. Mcmillan Publishers Limited, 2013.</i>
Bolt, B. and Hobbs, D.	<i>101 Mathematical Projects: A Resource Book.</i> United Kingdom: Cambridge University Press, 1994.
Bostock, L. and Chandler, S.	<i>Core Mathematics for Advanced-Levels (4th Ed.).</i> United Kingdom: <i>Oxford University Press</i> , 2015.
Campbell, E.	<i>Pure Mathematics for CAPE, Vol. 2. Jamaica: LMH Publishing Limited, 2007.</i>
Crawshaw, J. and Chambers, J.	<i>A Concise Course in A-Level Statistics.</i> Cheltenham, United Kingdom: Stanley Thornes (Publishers) Limited, 1999.
Mann, P.S.	<i>Introductory Statistics (9th Ed.).</i> New Jersey: Wiley, 2016.
Toolsie, Raymond	<i>Pure Mathematics: A Complete Course for CAPE Unit 2. Trinidad West Indies: Caribbean Educational Publishers Ltd, 2003.</i>

Websites:

<https://www.mathworks.com/help/matlab/examples/basic-matrix-operations.html>

◆ OUTLINE OF ASSESSMENT

Each Unit of the syllabus is assessed separately. The scheme of assessment for each Unit is the same. A candidate's performance on each Unit is reported as an overall grade and a grade on each Module of the Unit. The assessment comprises two components, one external and one internal.

EXTERNAL ASSESSMENT

(80 per cent)

The candidate is required to sit two written papers for a total of 4 hrs.

Paper 01 (1 hour 30 minutes)	This paper comprises forty-five, compulsory multiple-choice items, 15 from each module. Each item is worth 1 mark.	30 per cent
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Paper 02 (2 hours 30 minutes)	This paper comprises six, compulsory extended-response questions.	50 per cent
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SCHOOL-BASED ASSESSMENT

(20 per cent)

School-Based Assessment in respect of each Unit will contribute 20 percent to the total assessment of a candidate's performance on that Unit.

Paper 031 (20 per cent of the Total Assessment)

This paper is intended for candidates registered through schools or other approved institutions.

The School-Based Assessment comprises a project designed and internally assessed by the teacher and externally moderated by CXC®. This paper comprises a single project requiring candidates to demonstrate the practical application of Mathematics in everyday life. In essence, it should allow candidates to probe, describe and explain a mathematical area of interest and communicate the findings using mathematical symbols, language and tools. The topic chosen may be from any module or combination of different modules of the syllabus.

The project will primarily be theory based requiring the solution of a chosen problem, applying mathematical concepts and procedures from any Module in the syllabus in order to understand, describe or explain a real-world phenomenon. The candidate may include secondary data to support the theoretical concepts.

Paper 032 (Alternative to Paper 031) (2 hours)

This paper is an alternative to Paper 031, the School-Based Assessment and is intended for private candidates. This paper comprises three compulsory questions based on any of the modules or combination of different modules of the syllabus. The paper tests skills similar to those assessed in the School-Based Assessment.

MODERATION OF SCHOOL-BASED ASSESSMENT (PAPER 031)

School-Based Assessment Record Sheets are available online via the CXC®'s website www.cxc.org.

All School-Based Assessment Record of marks must be submitted online using the SBA data capture

module of the Online Registration System (ORS). Assignments will be requested by **CXC®** for moderation purposes. These assignments will be reassessed by **CXC®** Examiners who moderate the School-Based Assessment. Teachers' marks may be adjusted as a result of moderation. The Examiners' comments will be sent to schools. All *assignments* must be *submitted* by the stipulated deadlines.

Copies of the students' assignments that are not submitted must be retained by the school until three months after publication by **CXC®** of the examination results.

ASSESSMENT DETAILS FOR EACH UNIT

External Assessment by Written Papers (80 per cent of Total Assessment)

Paper 01 (1 hour 30 minutes – 30 per cent of Total Assessment)

1. Composition of the Paper

- (a) This paper consists of forty-five multiple-choice items, with fifteen items based on each Module.
- (b) All items are compulsory.

2. Syllabus Coverage

- (a) Knowledge of the entire syllabus is required.
- (b) The paper is designed to test a candidate's knowledge across the breadth of the syllabus.

3. Question Type

Questions may be presented using words, symbols, tables, diagrams or a combination of these.

4. Mark Allocation

- (a) Each item is allocated 1 mark.
- (b) Each Module is allocated 15 marks.
- (c) The total marks available for this paper is 45.
- (d) This paper contributes 30 per cent towards the final assessment.

5. Award of Marks

Marks will be awarded for conceptual knowledge, algorithmic knowledge and reasoning.

Conceptual Knowledge: Recall or selection of facts or principles.

Algorithmic Knowledge: Evidence of knowledge, ability to apply concepts and skills, and to analyse a problem in a logical manner; *computational skill, numerical accuracy, and acceptable tolerance limits in drawing diagrams.*

Reasoning: Selection of appropriate strategy, evidence of clear thinking, explanation and/or logical argument.

6. Use of Calculators

- (a) Each candidate is required to have a silent, non-programmable calculator for the duration of the examination, and is entirely responsible for its functioning.
- (b) The use of calculators with graphical displays will not be permitted.
- (c) Answers found by using a calculator, without relevant working shown, may not be awarded full marks.
- (d) Calculators must not be shared during the examination.

7. Use of Mathematical Tables

A booklet of mathematical formulae *and tables* will be provided.

Paper 02 (2 hours 30 minutes – 50 per cent of Total Assessment)

1. Composition of Paper

- (a) The paper consists of six questions. Two questions are based on each Module (Module 1, Module 2 and Module 3).
- (b) All questions are compulsory.

2. Syllabus Coverage

- (a) Each question may be based on one or more than one topic in the Module from which the question is taken.
- (b) Each question may develop a single theme or unconnected themes.

3. Question Type

- (a) Questions may require an extended response.
- (b) Questions may be presented using words, symbols, tables, diagrams or a combination of these.

4. Mark Allocation

- (a) Each question is worth 25 marks.

- (b) The number of marks allocated to each sub-question will appear in brackets on the examination paper.
- (c) Each Module is allocated 50 marks.
- (d) The total marks available for this paper is 150.
- (e) This paper contributes 50 per cent towards the final assessment.

5. Award of Marks

- (a) Marks will be awarded for conceptual knowledge, algorithmic knowledge and reasoning.

Conceptual Knowledge: Recall *and understand* facts or principles.

Algorithmic Knowledge: Evidence of knowledge, ability to apply concepts and skills, and to analyse a problem in a logical manner; *computational skill, numerical accuracy, and acceptable tolerance limits in drawing diagrams.*

Reasoning: Selection of appropriate strategy, evidence of clear thinking, explanation and/or logical argument.

- (b) Full marks will be awarded for **correct** answers and presence of **appropriate working**.
- (c) Where an incorrect answer is given, credit may be awarded for correct method provided that the working is shown.
- (d) If an incorrect answer in a previous question or part-question is used later in a section or a question, then marks may be awarded in the latter part even though the original answer is incorrect. In this way, a candidate is not penalised twice for the same mistake.
- (e) A correct answer given with no indication of the method used (in the form of written working) will receive no marks. Candidates are, therefore, advised to show all relevant working.

6. Use of Calculators

- (a) Each candidate is required to have a silent, non-programmable calculator for the duration of the examination, and is responsible for its functioning.
- (b) The use of calculators with graphical displays will not be permitted.
- (c) Answers found by using a calculator, without relevant working shown, may not be awarded full marks.
- (d) Calculators must not be shared during the examination.

7. Use of Mathematical Tables

A booklet of mathematical formulae *and tables* will be provided.

SCHOOL-BASED ASSESSMENT

School-Based Assessment is an integral part of student assessment in the course covered by this syllabus. It is intended to assist students in acquiring certain knowledge, skills, and attitudes that are associated with the subject. The activities for the School-Based Assessment are linked to the syllabus and should form part of the learning activities to enable the student to achieve the objectives of the syllabus. Group work should be encouraged.

During the course of study for the subject, students obtain marks for the competence they develop and demonstrate in undertaking their School-Based Assessment assignments. These marks contribute to the final marks and grades that are awarded to students for their performance in the examination.

*The guidelines provided in this syllabus for selecting appropriate tasks are intended to assist teachers and students in selecting assignments that are valid for the purpose of School-Based Assessment. The guidelines provided for the assessment of these assignments are intended to assist teachers in awarding marks that are reliable estimates of the achievement of students in the School-Based Assessment component of the course. In order to ensure that the scores awarded by teachers are in line with the **CXC**[®] standards, the Council undertakes the moderation of a sample of the School-Based Assessment assignments marked by each teacher.*

*School-Based Assessment provides an opportunity to individualise a part of the curriculum to meet the needs of students. It facilitates feedback to the student at various stages of the experience. This helps to build the self-confidence of students as they proceed with their studies. School-Based Assessment also facilitates the development of the critical skills and abilities emphasised by this **CAPE**[®] subject and enhance the validity of the examination on which candidate performance is reported. School-Based Assessment, therefore, makes a significant and unique contribution to both the development of relevant skills and the testing and rewarding of students for the development of those skills.*

The Caribbean Examinations Council seeks to ensure that the School-Based Assessment scores are valid and reliable estimates of accomplishment. The guidelines provided in this syllabus are intended to assist in doing so.

Paper 031 (20 per cent of the Total Assessment)

This paper is intended for candidates registered through schools or other approved institutions.

This paper comprises a project requiring candidates to demonstrate the practical application of Mathematics in everyday life. In essence, it should allow candidates to probe, describe and explain a mathematical area of interest and communicate the findings using mathematical symbols, language and tools.

The topic chosen will be determined largely by the candidate in consultation with the teacher and may be from any module or combination of different modules of the syllabus.

The project is based on applying mathematical concepts, skills and procedures from any topic(s) in order to understand, describe or explain a real world phenomenon. The project is theory based and no data collection is required. However, secondary data may be presented to support the theoretical concepts.

CRITERIA FOR THE SCHOOL-BASED ASSESSMENT (SBA) (Paper 031)

This paper is compulsory and consists of a project. Candidates have the option to work in small groups (maximum 5 members) to complete their SBAs.

1. The aims of the project are to:

- (a) promote self-learning;*
- (b) allow teachers the opportunity to engage in formative assessment of their students;*
- (c) provide opportunities for all candidates to show, with confidence, that they have mastered the syllabus;*
- (d) enable candidates to use the methods and procedures of statistical analysis to describe or explain real-life phenomena; and,*
- (e) foster the development of critical thinking skills among students.*

2. Requirements of the project

- (a) The project will be presented in the form of a report and should include the following:*
 - (i) a project title;*
 - (ii) a problem statement which provides the purpose of the project;*
 - (iii) identification of important elements of the problem;*
 - (iv) the solution of the problem; and,*
 - (v) discussion of findings.*
- (b) Teachers are expected to guide candidates in choosing appropriate projects that relate to their interests and mathematical expertise.*
- (c) Candidates should make use of mathematical skills and theories from any of the Modules.*

3. Integration of Project into the Course

- (a) The activities related to project work should be integrated into the course so as to enable candidates to learn and practise the skills of undertaking a successful project.*
- (b) Some time in class should be allocated for general discussion of project work. For example, discussion of which specific objectives should be selected, how the*

information should be presented and analysed.

- (c) *Class time should also be allocated for discussion between teacher and student, and student and student.*

4. Management of Project

(a) **Planning**

An early start to planning project work is highly recommended and the schedule of the dates for submission should be developed by teachers and candidates.

(b) **Length**

The project must not exceed 1,500 words. The word count does not include: Tables, References, Table of contents, Appendices and Figures. Two marks will be deducted for exceeding the word limit by more than 200 words.

(c) **Guidance**

Each candidate should know the requirements of the project and its assessment process.

Although candidates may consult with resource persons besides the teacher the candidates' submission should be their own work.

Candidates are not expected to work on their own. The teacher is expected to give appropriate guidance at all stages of project work, for example, chapters to read, alternative procedures to follow and other sources of information.

(d) **Authenticity**

Teachers are required to ensure that all projects are the candidates' work.

The recommended procedures are to:

- (i) *engage candidates in discussion;*
- (ii) *ask candidates to describe procedures used and summarise findings either orally or written; and,*
- (iii) *ask candidates to explain specific aspects of the analysis.*

◆ ASSESSMENT CRITERIA

The project will be presented in the form of a report and will be assessed on the following aspects:

- (a) *project title;*
- (b) *problem statement;*
- (c) *mathematical formulation;*
- (d) *the solution of the problem;*
- (e) *discussion of findings; and,*
- (f) *overall presentation.*

	Project Descriptors	Marks	Total
1.	Project Title		
	Title is clear and concise, and relates to real world problem Award 1 mark for titles presented and not related to real world problems.	2	
			2
2.	Problem Statement		
	Problem is clearly stated and is appropriate in level of difficulty	1	
	Purpose is clearly stated and relates to a real-world problem Award 1 mark for purpose presented and not related to real world problems.	2	
			3
3.	Mathematical Formulation		
	Identifies all the important elements of the problem and shows a complete understanding of the relationships between them	2	
	Shows understanding of the problem's mathematical concepts and principles	2	
	Uses appropriate mathematical terminology and notations to model the problem mathematically	2	
	Uses appropriate Mathematical methods for the problem/task	2	
			8
4.	The Problem Solution		
	Assumptions are clearly identified and explained	1	
	Proofs are clearly stated	1	
	Diagrams are appropriate and clearly labelled	2	
	Explanations are sufficient and clearly expressed	2	
	Theorems are appropriate and/or Formulae are relevant to the solutions and are correctly applied	2	
	Calculations are accurate without errors	2	
	Solutions are clearly stated	2	
	Interpretation of results is appropriate given the purpose	2	
	Applies the solution or proof to the given real-world problem	2	
			16
5.	Discussion of Findings		
	Discussion is coherent, concise and relates to the purpose of the project	2	
	Recommendations are relevant and practical	2	
	Conclusion is succinct, fully reflects the objectives and is supported by evidence	2	
			6
6.	Overall Presentation		
	Communicates information in a logical way using correct grammar, and appropriate mathematical jargon all the time.	2	
	Appropriate citing of sources	1	
	Inclusion of bibliography	2	
			5
	Total marks		40

Procedures for Reporting and Submitting School-Based Assessment

- (i) Teachers are required to record the mark awarded to each candidate on the mark sheet provided by **CXC**[®]. The completed mark sheets should be submitted to **CXC**[®] no later than April 30 of the year of the examination.

Note: The school is advised to keep a copy of the project for each candidate as well as copies of the mark sheets.

- (ii) Teachers will be required to submit to **CXC**[®] the projects of candidates according to the guidelines provided. The projects will be re-marked by **CXC**[®] for moderation purposes.

Moderation of School Based Assessment

The candidate's performance on the project will be moderated. The standard and range of marks awarded by the teacher will be adjusted where appropriate. However, the rank order assigned by the teacher will be adjusted only in special circumstances and then only after consideration of the data provided by the sample of marked projects submitted by the teacher and re-marked by **CXC**[®].

Paper 032 (20 per cent of Total Assessment)

1. Composition of Paper

- (a) This paper consists of three questions, based on *any of the modules or combination of different modules of the syllabus*.
- (b) All questions are compulsory.

2. Question Type

- (a) Each question may require an extended response.
- (b) A part of or an entire question may focus on mathematical modelling.
- (c) A question may be presented using words, symbols, tables, diagrams or a combination of these.

3. Mark Allocation

- (a) Each question carries a maximum of 20 marks.
- (b) The Paper carries a maximum of 60 marks.
- (c) For each question, marks should be allocated for the skills outlined on page 3 of this Syllabus.

4. Award of Marks

- (a) Marks will be awarded for reasoning, algorithmic knowledge and conceptual knowledge.

For each *question*, the 20 marks *will* be awarded as follows:

Conceptual Knowledge:	Recall <i>and understand</i> facts or principles.
Algorithmic Knowledge:	Evidence of knowledge, ability to apply concepts and skills, and to analyse a problem in a logical manner. <i>Computational skill, numerical accuracy, and acceptable tolerance limits in drawing diagrams.</i>
Reasoning:	Selection of appropriate strategy, evidence of clear reasoning, explanation and/or logical argument.

- (b) *Full marks will be awarded for correct answers and presence of appropriate working.*
- (c) *Where an incorrect answer is given, credit may be awarded for correct method provided that the working is shown.*
- (d) If an incorrect answer in a previous question or part-question is used later in a section or a question, then marks may be awarded in the later part even though the original answer is incorrect. In this way, a candidate is not penalised twice for the same mistake.
- (e) A correct answer given with no indication of the method used (in the form of written working) will receive no marks. Candidates should be advised to show all relevant working.

5. Use of Calculators

Each candidate is required to have a silent, non-programmable calculator for the duration of the examination, and is responsible for its functioning.

- (a) *The use of calculators with graphical displays will not be permitted.*
- (b) *Answers found by using a calculator, without relevant working shown, may not be awarded full marks.*
- (c) *Calculators must not be shared during the examination.*
- (d) *Cell phones should not be used as an alternative to calculators.*

6. Use of Mathematical Tables

A booklet of mathematical formulae and tables will be provided.

GENERAL GUIDELINES FOR TEACHERS

1. Teachers should note that the reliability of marks awarded is a significant factor in the *School-Based Assessment*, and has far-reaching implications for the candidate's final grade.
2. Candidates who do not fulfil the requirements of the *School-Based Assessment* will be considered absent from the whole examination.
3. Teachers are asked to note the following:
 - (a) the relationship between the marks for the assignment and those submitted to **CXC**[®] on the *School-Based Assessment* form should be clearly shown;
 - (b) *marks for the Project will be allocated across Modules in the ratio 1:1:1. The project will be marked out of a total of 40 marks. The marks earned by a student are assigned to each Module. For example, if a student earns 35 out of 40 for his School-Based Assessment, 35 marks will be assigned to Module 1, 35 marks to Module 2 and 35 marks to Module 3. The total score will be $35+35+35=105$ out of 120; and,*
 - (c) the standard of marking should be consistent.

◆ REGULATIONS FOR PRIVATE CANDIDATES



Candidates who are registered privately will be required to sit Paper 01, Paper 02 and Paper 032. Paper 032 *takes the form of a written examination and* will be 2 hours' duration and will consist of three questions, each worth 20 marks. Each question will be based on the objectives and content of one of the three Modules of the Unit. Paper 032 will contribute 20 per cent of the total assessment of a candidate's performance on that Unit *and* will test the same skills as the School-Based Assessment.

Paper 032 (2 hours)

The paper consists of three questions. Each question *based on any of the modules or combination of different modules of the syllabus* and tests candidates' skills and abilities to:

1. recall, select and use appropriate facts, concepts and principles in a variety of contexts;
2. manipulate mathematical expressions and procedures using appropriate symbols and language, logical deduction and inferences;
3. select and use a simple mathematical model to describe a real-world situation;
4. simplify and solve mathematical models; and,
5. interpret mathematical results and their application in a real-world problem.

◆ REGULATIONS FOR RESIT CANDIDATES



CAPE® candidates may reuse any moderated SBA score within a two-year period. In order to assist candidates in making decisions about whether or not to reuse a moderated SBA score, the Council will continue to indicate on the preliminary results if a candidate's moderated SBA score is less than 50 per cent in a particular Unit. Candidates re-using SBA scores should register as "Re-sit candidates" and must provide the previous candidate number when registering.

◆ ASSESSMENT GRID

The Assessment Grid for each Unit contains marks assigned to papers and to Modules and percentage contributions of each paper to total scores.

Units 1 and 2

Papers	Module 1	Module 2	Module 3	Total	(%)
External Assessment Paper 01 (1 hour 30 minutes)	15 (30 weighted)	15 (30 weighted)	15 (30 weighted)	45 (90 weighted)	(30)
Paper 02 (2 hours 30 minutes)	50	50	50	150	(50)
School-Based Assessment Paper 031 or Paper 032 (2 hours)	40 (20 weighted)	40 (20 weighted)	40 (20 weighted)	120 (60 weighted)	(20)
Total	100	100	100	300	(100)

◆ MATHEMATICAL NOTATION

Set Notation

$\in$	is an element of
$\notin$	is not an element of
$\{x: \dots\}$	the set of all x such that ...
$n(A)$	the number of elements in set A
$\emptyset$	the empty set
U	the universal set
A'	the complement of the set A
$\mathbb{W}$	the set of whole numbers $\{0, 1, 2, 3, \dots\}$
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{I}$	<i>the set of irrational numbers</i>
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\subset$	is a <i>proper</i> subset of
$\not\subset$	is not a proper subset of
$\subseteq$	<i>is a subset of</i>
$\not\subseteq$	is not a subset of
$\cup$	union
$\cap$	intersection
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leq x \leq b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leq x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leq b\}$

Logic

$\wedge$	conjunction
$\vee$	(inclusive) injunction
$\underline{\vee}$	exclusive disjunction
$\neg$	negation
$\rightarrow$	conditionality
$\leftrightarrow$	bi-conditionality
$\Rightarrow$	implication
$\Leftrightarrow$	equivalence

Miscellaneous Symbols

$\equiv$	is identical to
$\approx$	is approximately equal to
$\propto$	is proportional to
∞	infinity

The following list summarises the notation used in the Mathematics papers of the Caribbean Advanced Proficiency Examination.

Operations

$\sum_{i=1}^n x_i$	$x_1 + x_2 + \dots + x_n$
$\sqrt{x}$	the positive square root of the real number x
$ x $	the modulus of the real number x
$n!$	n factorial, $1 \times 2 \times \dots \times n$ for $n \in \mathbb{N}$ ($0! = 1$)
${}^nC_r \binom{n}{r}$	the binomial coefficient, $\frac{n!}{(n-r)!r!}$, for $n, r \in \mathbb{N}$, $0 \leq r \leq n$
${}^nP_r$	$\frac{n!}{(n-r)!}$

Functions

f	the function f
$f(x)$	the value of the function f at x
$f: A \rightarrow B$	the function f under which each element of the set A has an image in the set B
$f: x \rightarrow y$	the function f maps the element x to the element y
f^{-1}	the inverse of the function f
Fg	the composite function $f(g(x))$
$\lim_{x \rightarrow a} f(x)$	the limit of $f(x)$ as x tends to a
$\Delta x, \delta x$	an increment of x
$\frac{dy}{dx}, y'$	the <i>first</i> derivative of y with respect to x
$\frac{d^n y}{dx^n}, y^{(n)}$	the n th derivative of y with respect to x
$f'(x), f''(x), \dots, f^{(n)}(x)$	the first, second, ..., n th derivatives of $f(x)$ with respect to x
$\dot{x}, \ddot{x}$	the first and second derivatives of x with respect to time t
e	the exponential constant
$\ln x$	the natural logarithm of x (to base e)
$\lg x$	the logarithm of x to base 10

Complex Numbers

i	$\sqrt{-1}$
z	a complex number, $z = x + yi$ where $x, y \in \mathbb{R}$
$\operatorname{Re} z$	the real part of z

$\operatorname{Im} z$	the imaginary part of z
$ z $	the modulus of z
$\arg z$	the argument of z , where $-\pi < \arg z \leq \pi$
$\bar{z}, z^*$	the complex conjugate of z

Vectors

$\underline{a}, \mathbf{a}, \overline{AB}$	vectors
$\hat{\mathbf{a}}$	a unit vector in the direction of the vector $\mathbf{a}$
$ \mathbf{a} $	the magnitude of the vector $\mathbf{a}$
$\mathbf{a} \cdot \mathbf{b}$	the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the positive Cartesian coordinate axes
$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Probability

S	the sample space
$A, B, \dots$	the events $A, B, \dots$
$P(A')$	the probability that the event A <i>does not occur</i>

Matrices

M	a matrix M
(M^{-1})	inverse of the non-singular square matrix M
M^T, M_T	transpose of the matrix M
$\det M, M $	determinant of the square matrix M

◆ GLOSSARY OF EXAMINATION TERMS

WORD	DEFINITION	NOTES
Analyse	examine in detail	
Annotate	add a brief note to a label	Simple phrase or a few words only.
Apply	use knowledge/principles to solve problems	Make inferences/conclusions.
Assess	present reasons for the importance of particular structures, relationships or processes	Compare the advantages and disadvantages or the merits and demerits of a particular structure, relationship or process.
Calculate	arrive at the solution to a numerical problem	Steps should be shown; units must be included.
Classify	divide into groups according to observable characteristics	
Comment	state opinion or view with supporting reasons	
Compare	state similarities and differences	An explanation of the significance of each similarity and difference stated may be required for comparisons which are other than structural.
Construct	use a specific format to make and/or draw a graph, histogram, pie chart or other representation using data or material provided or drawn from practical investigations, build (for example, a model), draw scale diagram	Such representations should normally bear a title, appropriate headings and legend.
Deduce	make a logical connection between two or more pieces of information; use data to arrive at a conclusion	

WORD	DEFINITION	NOTES
Define	state concisely the meaning of a word or term	This should include the defining equation/formula where relevant.
Demonstrate	show; direct attention to...	
Derive	to deduce, determine or extract from data by a set of logical steps some relationship, formula or result	This relationship may be general or specific.
Describe	provide detailed factual information of the appearance or arrangement of a specific structure or a sequence of a specific process	Description may be in words, drawings or diagrams or any appropriate combination. Drawings or diagrams should be annotated to show appropriate detail where necessary.
Determine	find the value of a physical quantity	
Design	plan and present with appropriate practical detail	Where hypotheses are stated or when tests are to be conducted, possible outcomes should be clearly stated and/or the way in which data will be analysed and presented.
Develop	expand or elaborate an idea or argument with supporting reasons	
Diagram	simplified representation showing the relationship between components	
Differentiate/Distinguish (between/among)	state or explain briefly those differences between or among items which can be used to define the items or place them into separate categories	
Discuss	present reasoned argument; consider points both for and against; explain the relative merits of a case	
Draw	make a line representation from specimens or apparatus which shows an accurate relation between the parts	In the case of drawings from specimens, the magnification must always be stated.

WORD	DEFINITION	NOTES
Estimate	make an approximate quantitative judgement	
Evaluate	weigh evidence and make judgements based on given criteria	The use of logical supporting reasons for a particular point of view is more important than the view held; usually both sides of an argument should be considered.
Explain	give reasons based on recall; account for	
Find	locate a feature or obtain as from a graph	
Formulate	devise a hypothesis	
Identify	name or point out specific components or features	
Illustrate	show clearly by using appropriate examples or diagrams, sketches	
Interpret	explain the meaning of	
Investigate	use simple systematic procedures to observe, record data and draw logical conclusions	
Justify	explain the correctness of	
Label	add names to identify structures or parts indicated by pointers	
List	itemise without detail	
Measure	take accurate quantitative readings using appropriate instruments	
Name	give only the name of	No additional information is required.
Note	write down observations	

WORD	DEFINITION	NOTES
Observe	pay attention to details which characterise a specimen, reaction or change taking place; to examine and note scientifically	Observations may involve all the senses and/or extensions of them but would normally exclude the sense of taste.
Outline	give basic steps only	
Plan	prepare to conduct an investigation	
Predict	use information provided to arrive at a likely conclusion or suggest a possible outcome	
Record	write an accurate description of the full range of observations made during a given procedure	This includes the values for any variable being investigated; where appropriate, recorded data may be depicted in graphs, histograms or tables.
Relate	show connections between; explain how one set of facts or data depend on others or are determined by them	
Sketch	make a simple freehand diagram showing relevant proportions and any important details	
State	provide factual information in concise terms outlining explanations	
Suggest	offer an explanation deduced from information provided or previous knowledge. (... a hypothesis; provide a generalisation which offers a likely explanation for a set of data or observations.)	No correct or incorrect solution is presumed but suggestions must be acceptable within the limits of scientific knowledge.
Use	apply knowledge/principles to solve problems	Make inferences/conclusions.

◆ GLOSSARY OF MATHEMATICAL TERMS

WORDS	MEANING
Absolute Value	The absolute value of a real number x , denoted by $ x $, is defined by $ x = x$ if $x > 0$ and $ x = -x$ if $x < 0$. For example, $ -4 = 4$.
Algorithm	A process consisting of a specific sequence of operations to solve a certain types of problems. See Heuristic .
Argand Diagram	An Argand diagram is a rectangular coordinate system where the complex number $x + iy$ is represented by the point whose coordinates are x and y . The x -axis is called the real axis and the y -axis is called the imaginary axis.
Argument of a Complex Number	The angle, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, is called the argument of a complex number $z = x + iy$.
Arithmetic Mean	The average of a set of values found by dividing the sum of the values by the amount of values.
Arithmetic Progression	An arithmetic progression is a sequence of elements, $a_1, a_2, a_3, \dots$, such that there is a common difference of successive terms. For example, the sequence $\{2, 5, 8, 11, 14, \dots\}$ has common difference, $d = 3$.
Asymptotes	A straight line is said to be an asymptote of a curve if the curve has the property of becoming and staying arbitrarily close to the line as the distance from the origin increases to infinity.
Augmented Matrix	If a system of linear equations is written in matrix form $Ax = b$, then the matrix $[A b]$ is called the augmented matrix.
Average	The average of a set of values is the number which represents the usual or typical value in that set. Average is synonymous with measures of central tendency. These include the mean, mode and median.
Axis of symmetry	A line that passes through a figure such that the portion of the figure on one side of the line is the mirror image of the portion on the other side of the line.
Bar Chart	A bar chart is a diagram which is used to represent the frequency of each category of a set of data in such a way that the height of each bar is proportionate to the frequency of the category it represents. Equal space should be left between consecutive bars to indicate it is not a histogram.

WORDS	MEANING
Base	In the equation $y = \log_a x$, the quantity a is called the base. The base of a polygon is one of its sides; for example, a side of a triangle. The base of a solid is one of its faces; for example, the flat face of a cylinder. The base of a number system is the number of digits it contains; for example, the base of the binary system is two.
Bias	Bias is systematically misestimating the characteristics of a population (parameters) with the corresponding characteristics of the sample (statistics).
Biased Sample	A biased sample is a sample produced by methods which ensures that the statistics is systematically different from the corresponding parameters.
Bijjective	A function is bijective if it is both injective and surjective; that is, both one-to-one, into and onto.
Bimodal	Bimodal refers to a set of data with two equally common modes.
Binomial	An algebraic expression consisting of the sum or difference of two terms. For example, $(ax + b)$ is a binomial.
Binomial Coefficients	The coefficients of the expansion $(x + y)^n$ are called binomial coefficients. For example, the coefficients of $(x + y)^3$ are 1, 3, 3 and 1.
Box-and-whiskers Plot	A box-and-whiskers plot is a diagram which displays the distribution of a set of data using the five number summary. Lines perpendicular to the axis are used to represent the five number summary. Single lines parallel to the x-axis are used to connect the lowest and highest values to the first and third quartiles respectively and double lines parallel to the x-axis form a box with the inner three values.
Categorical Variable	A categorical variable is a variable measured in terms possession of quality and not in terms of quantity.
Class Intervals	Non-overlapping intervals, which together contain every piece of data in a survey.
Closed Interval	A closed interval is an interval that contains its end points; it is denoted with square brackets $[a, b]$. For example, the interval $[-1, 2]$ contains -1 and 2 . For contrast see open interval .

WORDS	MEANING
Composite Function	A function consisting of two or more functions such that the output of one function is the input of the other function. For example, in the composite function $f(g(x))$ the input of f is g .
Compound Interest	A system of calculating interest on the sum of the initial amount invested together with the interest previously awarded; if A is the initial sum invested in an account and r is the rate of interest per period invested, then the total after n periods is $A(1 + r)^n$.
Combinations	The term combinations refers to the number of possible ways of selecting r objects chosen from a total sample of size n if you don't care about the order in which the objects are arranged. Combinations are calculated using the formula ${}^nC_r = \frac{n!}{r!(n-r)!}$. See factorial .
Complex Numbers	A complex number is formed by adding a pure imaginary number to a real number. The general form of a complex number is $z = x + iy$, where x and y are both real numbers and i is the imaginary unit: $i^2 = -1$. The number x is called the real part of the complex number, while the number y is called the imaginary part of the complex number.
Conditional Probability	The conditional probability is the probability of the occurrence of one event affecting another event. The conditional probability of event A occurring given that event B has occurred is denoted $P(A B)$ (read "probability of A given B "). The formula for conditional probability is $P(A B) = \frac{P(A \cap B)}{P(B)}$
Conjugate of a Complex Number	The conjugate of a complex number $z = x + iy$ is the complex number $\bar{z} = x - iy$, found by changing the sign of the imaginary part. For example, if $z = 3 - 4i$, then $\bar{z} = 3 + 4i$.
Continuous	The graph of $y = f(x)$ is continuous at a point a if: $f(a)$ exists, $\lim_{x \rightarrow a} f(x)$ exists, and $\lim_{x \rightarrow a} f(x) = f(a)$. A function is said to be continuous in an interval if it is continuous at each point in the interval.
Continuous Random Variable	A continuous random variable is a random variable that can take on any real number value within a specified range. For contrast, see Discrete Random Variable .

WORDS	MEANING
Coterminal	Two angles are said to be coterminal if they have the same initial and terminal arms. For example, $\theta = 30^\circ$ is coterminal with $\alpha = 390^\circ$.
Critical Point	A critical point of a function $f(x)$ is the point $P(x,y)$ where the first derivative, $f'(x)$ is zero. See also stationary points .
Data	Data (plural of datum) are the facts about something. For example, the height of a building.
Degree	1. The degree is a unit of measure for angles. One degree is $\frac{1}{360}$ of a complete rotation. See also Radian. 2. The degree of a polynomial is the highest power of the variable that appears in the polynomial. For example, the polynomial $p(x) = 2 + 3x - x^2 + 7x^3$ has degree 3.
Delta	The Greek capital letter delta, which has the shape of a triangle: Δ , is used to represent "change in". For example Δx represents "change in x ".
Dependent Events	In Statistics, two events A and B are said to be dependent if the occurrence of one event affects the probability of the occurrence of the other event. For contrast, see Independent Events .
Derivative	The derivative of a function $y = f(x)$ is the rate of change of that function. The notations used for derivative include: $y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$
Descriptive Statistics	Descriptive statistics refers to a variety of techniques that allows for general description of the characteristics of the data collected. It also refers to the study of ways to describe data. For example, the mean, median, variance and standard deviation are descriptive statistics. For contrast, see Inferential Statistics .
Determinant	The determinant of a matrix is a number that is useful for describing the characteristics of the matrix. For example if the determinant is zero then the matrix has no inverse.
Differentiable	A continuous function is said to be differentiable over an interval if its derivative exists for every point in that interval. That means that the graph of the function is smooth with no kinks, cusps or breaks.
Differential Equation	A differential equation is an equation involving the derivatives of a function of one or more variables. For example, the equation $\frac{dy}{dx} - y = 0$ is a differential equation.

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Differentiation	Differentiation is the process of finding the derivative.
Discrete	A set of values are said to be discrete if they are all distinct and separated from each other. For example the set of shoe sizes where the elements of this set can only take on a limited and distinct set of values. See Discrete Random Variables .
Discrete Random Variable	A discrete random variable is a random variable that can only take on values from a discrete list. For contrast, see Continuous Random Variables .
Estimate	The best guess for an unknown quantity arrived at after considering all the information given in a problem.
Even Function	A function $y = f(x)$ is said to be even if it satisfies the property that $f(x) = f(-x)$. For example, $f(x) = \cos x$ and $g(x) = x^2$ are even functions. For contrast, see Odd Function.
Event	In probability, an event is a set of outcomes of an experiment. For example, the <i>event</i> A may be defined as obtaining two heads from tossing a coin twice.
Expected Value	The average amount that is predicted if an experiment is repeated many times. The expected value of a random variable X is denoted by $E[X]$. The expected value of a discrete random variable is found by taking the sum of the product of each outcome and its associated probability. In short, $E[X] = \sum_{i=1}^n x_i p(x_i).$
Experimental Probability	Experimental probability is the <i>chance</i> of something happening, based on repeated testing and observing results. It is the ratio of the number of times an event occurred to the number of times tested. For example, to find the experimental probability of winning a game, one must play the game many times, then <i>divide the number</i> of games won by the total number of games played.
Exponent	An exponent is a symbol or a number written above and to the right of another number. It indicates the operation of repeated multiplication.
Exponential Function	A function that has the form $y = a^x$, where a is any real number and a is called the base.
Extrapolation	An extrapolation is a predicted value that is outside the range of previously observed values. For contrast, see Interpolation .
Factor	A factor is one of two or more expressions which are multiplied together. A prime factor is an indecomposable factor. For example, the

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factors of $(x^2 - 4)(x + 3)$ include $(x^2 - 4)$ and $(x + 3)$, where $(x + 3)$ is prime but $(x^2 - 4)$ is not prime as it can be further decomposed into $(x - 2)(x + 2)$.

Factorial	The factorial of a positive integer n is the product of all the integers from 1 up to n and is denoted by $n!$, where $1! = 0! = 1$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.
Function	A correspondence in which each member of one set is mapped unto a member of another set.
Geometric Progression	A geometric progression is a sequence of terms obtained by multiplying the previous term by a fixed number which is called the common ratio. A geometric progression is of the form $a, ar, ar^2, ar^3, \dots$
Graph	A visual representation of data that displays the relationship among variables, usually cast along x and y axes.
Grouped Data	Grouped data refers to a range of values which are combined together so as to make trends in the data more apparent.
Heterogeneity	Heterogeneity is the state of being of incomparable magnitudes. For contrast, see Homogeneity .
Heuristic	A heuristic method of solving problems involve intelligent trial and error. For contrast, see Algorithm .
Histogram	A histogram is a bar graph with no spaces between the bars where the area of the bars is proportionate to the corresponding frequencies. If the bars have the same width then the heights are proportionate to the frequencies.
Homogeneity	Homogeneity is the state of being of comparable magnitudes. For contrast, see Heterogeneity .
Identity	1. An equation that is true for every possible value of the variables. For example $x^2 - 1 \equiv (x - 1)(x + 1)$ is an identity while $x^2 - 1 = 3$ is not, as it is only true for the values $x = \pm 2$. 2. The identity element of an operation is a number such that when operated on with any other number results in the other number. For example, the identity element under addition of real numbers is zero; the identity element under multiplication of 2×2 matrices is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

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Independent Events	In Statistics, two events are said to be independent if they do not affect each other. That is, the occurrence of one event does not depend on whether or not the other event occurred.
Inferential Statistics	Inferential Statistics is the branch of mathematics which deals with the generalisations of samples to the population of values.
Infinity	The symbol ∞ indicating a limitless quantity. For example, the result of a nonzero number divided by zero is infinity.
Integration	Integration is the process of finding the integral which is the antiderivative of a function.
Interpolation	An interpolation is an estimate of an unknown value which is within the range of previously observed values. For contrast, see Extrapolation .
Interval	An interval on a number line is a continuum of points bounded by two limits (end points). An Open Interval refers to an interval that excludes the end points and is denoted (a, b). For example, $(0, 1)$. A Closed Interval is an interval which includes the end points and is denoted $[a, b]$. For example $[-1, 3]$. A Half-Open Interval is an interval which includes one end point and excludes the other. For example, $[0, \infty)$.
Interval Scale	Interval scale refers to data where the difference between values can be quantified in absolute terms and any zero value is arbitrary. Finding a ratio of data values on this scale gives meaningless results. For example, temperature is measured on the interval scale: the difference between 19°C and 38°C is 19°C , however, 38°C is not twice as warm as 19°C and a temperature of 0°C does not mean there is no temperature. See also Nominal, Ordinal and Ratio scales.
Inverse	1. The inverse of an element under an operation is another element which when operated on with the first element results in the identity. For example, the inverse of a real number under addition is the negative of that number. 2. The inverse of a function $f(x)$ is another function denoted $f^{-1}(x)$, which is such that $f[f^{-1}(x)] = f^{-1}[f(x)] = x$.
Irrational Number	A number that cannot be represented as an exact ratio of two integers. For example, π or the square root of 2.
Limit	The limit of a function is the value which the dependent variable approaches as the independent variable approaches some fixed value.

WORDS	MEANING
Line of Best Fit	The line of best fit is the line that minimises the sum of the squares of the deviations between each point and the line.
Linear Expression	An expression of the form $ax + b$ where x is a variable and a and b are constants, or in more variables, an expression of the form $ax + by + c, ax + by + cz + d$ where a, b, c and d are constants.
Logarithm	A logarithm is the power of another number called the base that is required to make its value a third number. For example 3 is the logarithm which carries 2 to 8. In general, if y is the logarithm which carries a to x, then it is written as $y = \log_a x$ where a is called the base. There are two popular bases: base 10 and base e. 1. The Common Logarithm (log): the equation $y = \log x$ is the shortened form for $y = \log_{10} x$. 2. The Natural Logarithm (ln): The equation $y = \ln x$ is the shortened form for $y = \log_e x$.
Matrix	A rectangular arrangement of numbers in rows and columns.
Method	In Statistics, the research methods are the tools, techniques or processes that we use in our research. These might be, for example, surveys, interviews, or participant observation. Methods and how they are used are shaped by methodology.
Methodology	Methodology is the study of how research is done, how we find out about things, and how knowledge is gained. In other words, methodology is about the principles that guide our research practices. Methodology therefore explains why we're using certain methods or tools in our research.
Modulus	The modulus of a complex number $z = x + iy$ is the real number $ z = \sqrt{x^2 + y^2}$. For example, the modulus of $z = -7 + 24i$ is $ z = \sqrt{(-7)^2 + 24^2} = 25$
Mutually Exclusive Events	Two events are said to be mutually exclusive if they cannot occur simultaneously, in other words, if they have nothing in common. For example, the event "Head" is mutually exclusive to the event "Tail" when a coin is tossed.
Mutually Exhaustive Events	Two events are said to be mutually exhaustive if their union represents the sample space.
Nominal Scale	Nominal scale refers to data which names of the outcome of an experiment. For example, the country of origin of the members of the West Indies cricket team. See also Ordinal, Interval and Ratio scales.

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Normal	The normal to a curve is a line which is perpendicular to the tangent to the curve at the point of contact.
Odd Function	A function is an odd function if it satisfies the property that $f(-x) = -f(x)$. For example, $f(x) = \sin x$ and $g(x) = x^3$ are odd functions. For contrast, see Even Function .
Ordinal Scale	Data is said to be in the ordinal scale if they are names of outcomes where sequential values are assigned to each name. For example, if Daniel is ranked number 3 on the most prolific goal scorer at the Football World Cup, then it indicates that two other players scored more goals than Daniel. However, the difference between the 3 rd ranked and the 10 th ranked is not necessarily the same as the difference between the 23 rd and 30 th ranked players. See also Nominal, Interval and Ratio scales .
Outlier	An outlier is an observed value that is significantly different from the other observed values.
Parameter	In statistics, a parameter is a value that characterises a population.
Partial Derivative	The partial derivative of $y = f(x_1, x_2, x_3, \dots, x_n)$ with respect to x_i is the derivative of y with respect to x , while all other independent variables are treated as constants. The partial derivative is denoted by $\frac{\partial f}{\partial x}$. For example, if $f(x, y, z) = 2xy + x^2z - \frac{3x^3y}{z}$, then $\frac{\partial f}{\partial x} = 2y + 2xz - \frac{9x^2y}{z}$.
Pascal Triangle	The Pascal triangle is a triangular array of numbers such that each number is the sum of the two numbers above it (one left and one right). The numbers in the n^{th} row of the triangle are the coefficients of the binomial expansion $(x + y)^n$.
Percentile	The p^{th} percentile of a list of numbers is the smallest value such that $p\%$ of the numbers in the list is below that value. See also Quartiles .
Permutations	Permutations refers to the number of different ways of selecting a group of r objects from a set of n objects when the order of the elements in the group is of importance and the items are not replaced. If $r = n$ then the permutations is $n!$; if $r < n$ then the number of permutation is ${}^n P_r = \frac{n!}{(n-r)!}$.
Piecewise Continuous Function	A function is said to be piecewise continuous if it can be broken into different segments where each segment is continuous.
Polynomial	A polynomial is an algebraic expression involving a sum of algebraic terms with nonnegative integer powers. For example, $2x^3 + 3x^2 - x + 6$ is a polynomial in one variable.

WORDS	MEANING
Population	In statistics, a population is the set of all items under consideration.
Principal Root	The principal root of a number is the positive root. For example, the principal square root of 36 is 6 (not -6).
Principal Value	The principal value of the arcsin and arctan functions lies on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The principal value of the arcos function lies on the interval $0 \leq x \leq \pi$.
Probability	1. The probability of an event is a measure of how likely it is for an event to occur. The probability of an event is always a number between zero and 1. 2. Probability is the study of chance occurrences.
Probability Distribution	A probability distribution is a table or function that gives all the possible values of a random variable together with their respective probabilities.
Probability Space	The probability space is the set of all outcomes of a probability experiment.
Proportion	1. A relationship between two ratios in which the first ratio is always equal to the second. Usually of the form $\frac{a}{b} = \frac{c}{d}$. 2. The fraction of a part and the whole. If two parts of a whole are in the ratio 2:7, then the corresponding proportions are $\frac{2}{9}$ and $\frac{7}{9}$ respectively.
Pythagorean Triple	A Pythagorean triple refers to three numbers, a , b & c , satisfying the property that $a^2 + b^2 = c^2$.
Quadrant	The four parts of the coordinate plane divided by the x - and y -axes are called quadrants. Each of these quadrants has a number designation. First quadrant – contains all the points with positive x and positive y coordinates. Second quadrant – contains all the points with negative x and positive y coordinates. The third quadrant contains all the points with both coordinates negative. Fourth quadrant – contains all the points with positive x and negative y coordinates.
Quadrantal Angles	Quadrantal Angles are the angles measuring 0° , 90° , 180° & 270° and all angles coterminal with these. See Coterminal .
Quartic	A quartic equation is a polynomial of degree 4.
Quartiles	Consider a set of numbers arranged in ascending or descending order. The quartiles are the three numbers which divide the set into four parts of equal amount of numbers. The first quartile in a list of

WORDS**MEANING**

numbers is the number such that a quarter of the numbers is below it. The second quartile is the median. The third quartile is the number such that three quarters of the numbers are below it. See also **Percentile**.

Quintic

A quintic equation is a polynomial of degree 5.

Radian

The radian is a unit of measure for angles, where one radian is $\frac{1}{2\pi}$ of a complete rotation. One radian is the angle in a circle subtended by an arc of length equal to that of the radius of the circle. See also **Degrees**.

Radical

The radical symbol ($\sqrt{}$) is used to indicate the taking of a root of a number. $\sqrt[q]{x}$ means the q^{th} root of x ; if $q = 2$ then it is usually written as $\sqrt{x}$. For example $\sqrt[5]{243} = 3$, $\sqrt[4]{16} = 2$. The radical always means to take the positive value. For example, both 5 and -5 satisfy the equation $x^2 = 25$, but $\sqrt{25} = 5$.

Random Variable

A random variable is a variable that takes on a particular value when a random event occurs.

Ratio Scale

Data are said to be on the ratio scale if they can be ranked, the distance between two values can be measured and the zero is absolute, that is, zero means "absence of". See also Nominal, Ordinal and Interval Scales.

Regression

Regression is a statistical technique used for determining the relationship between two quantities.

Residual

In linear regression, the residual refers to the difference between the actual point and the point predicted by the regression line. That is the vertical distance between the two points.

Root

1. The root of an equation is the same as the solution of that equation. For example, if $y = f(x)$, then the roots are the values of x for which $y = 0$. Graphically, the roots are the x -intercepts of the graph.
2. The n^{th} root of a real number x is a number which, when multiplied by itself n times, gives x . If n is odd then there is one root for every value of x ; if n is even then there are two roots (one positive and one negative) for positive values of x and no real roots for negative values of x . The positive root is called the **Principal root** and is represented by the radical sign ($\sqrt{}$). For example, the principal square root of 9 is written as $\sqrt{9} = 3$ but the square roots of 9 are $\pm\sqrt{9} = \pm 3$.

Sample

A group of items chosen from a population.

WORDS	MEANING
Sample Space	The set of all possible outcomes of a probability experiment. Also called probability space.
Sampling Frame	In statistics, the sampling frame refers to the list of cases from which a sample is to be taken.
Scientific Notation	A shorthand way of writing very large or very small numbers. A number expressed in scientific notation is expressed as a decimal number between 1 and 10 multiplied by a power of 10 (for example, $7000 = 7 \times 10^3$ or $0.0000019 = 1.9 \times 10^{-6}$).
Series	A series is an indicated sum of a sequence.
Sigma	1. The Greek capital letter sigma, Σ, denotes the summation of a set of values. 2. The corresponding lowercase letter sigma, σ, denotes the standard deviation.
Significant Digits	The amount of digits required for calculations or measurements to be close enough to the actual value. Some rules in determining the number of digits considered significant in a number: - The leftmost non-zero digit is the first significant digit. - Zeros between two non-zero digits are significant. - Trailing zeros to the right of the decimal point are considered significant.
Simple Event	A non-decomposable outcome of a probability experiment.
Skew	Skewness is a measure of the asymmetry of a distribution of data.
Square Matrix	A matrix with equal number of rows and columns.
Square Root	The square root of a positive real number n is the number m such that $m^2 = n$. For example, the square roots of 16 are 4 and -4.
Standard Deviation	The standard deviation of a set of numbers is a measure of the average deviation of the set of numbers from their mean.
Stationary Point	The stationary point of a function $f(x)$ is the point $P(x_0, y_0)$ where $f'(x) = 0$. There are three type of stationary points, these are: 1. Maximum point is the stationary point such that $\frac{d^2f}{dx^2} < 0$; 2. Minimum point is the stationary point such that $\frac{d^2f}{dx^2} > 0$; 3. Point of Inflexion is the stationary point where $\frac{d^2f}{dx^2} = 0$ and the point is neither a maximum nor a minimum point.

WORDS	MEANING
Statistic	A statistic is a quantity calculated from among the set of items in a sample.
Statistical Inference	The process of estimating unobservable characteristics of a population by using information obtained from a sample.
Symmetry	Two points A and B are symmetric with respect to a line if the line is a perpendicular bisector of the segment AB .
Tangent	A line is a tangent to a curve at a point A if it just touches the curve at A in such a way that it remains on one side of the curve at A . A tangent to a circle intersects the circle only once.
Theoretical Probability	The chances of events happening as determined by calculating results that would occur under ideal circumstances. For example, the theoretical probability of rolling a 4 on a fair four-sided die is $\frac{1}{4}$ or 25%, because there is one chance in four to roll a 4, and under ideal circumstances one out of every four rolls would be a 4.
Trigonometry	The study of triangles. Three trigonometric functions defined for either acute angles in the right-angled triangle are: Sine of the angle x is the ratio of the side opposite the angle and the hypotenuse. In short, $\sin x = \frac{O}{H}$; Cosine of the angle x is the ratio of the short side adjacent to the angle and the hypotenuse. In short, $\cos x = \frac{A}{H}$; Tangent of the angle x is the ratio of the side opposite the angle and the short side adjacent to the angle. In short $\tan x = \frac{O}{A}$.
Z-Score	The z-score of a value x is the number of standard deviations it is away from the mean of the set of all values, $z_{score} = \frac{x - \bar{x}}{\sigma}$.

◆ ADDITIONAL NOTES

UNIT 1

MODULE 1: BASIC ALGEBRA AND FUNCTIONS

Proof by Mathematical Induction (MI)

Typical Question

Prove that some formula or statement P_n is true for all positive integers $n \in k$, where k is some positive integer; usually $k = 1$.

Procedure

Step 1: Verify that when $k = 1$: P_n is true for $n = k = 1$. This establishes that P_n is true for $n = 1$.

Step 2: Assume P_n is true for $n = k$, where k is a positive integer > 1 . At this point, the statement k replaces n in the statement P_n and is taken as true.

Step 3: Show that P_n is true for $n = k + 1$ using the true statement in step 2 with n replaced by $k + 1$.

Step 4: At the end of step 3, it is stated that statement P_n is true for all positive integers $n \geq k$.

Summary

Proof by MI: For $k > 1$, verify Step 1 for k and proceed through to Step 4.

Observation

Most users of MI do not see how this proves that P_n is true. The reason for this is that there is a massive gap between Steps 3 and 4 which can only be filled by becoming aware that Step 4 only follows because Steps 1 to 3 are repeated an infinity of times to generate the set of all positive integers. The focal point is the few words “for all positive integers $n \geq k$ ” which points to the determination of the set S of all positive integers for which P is true.

Step 1 says that $1 \in S$ for $k = 1$.

Step 3 says that $k + 1 \in S$ whenever $k \in S$, so immediately $2 \in S$ since $1 \in S$.

Iterating on Step 3 says that $3 \in S$ since $2 \in S$ and so on, so that $S = \{1, 2, 3, \dots\}$, that is, S is the set of all positive integers when $k = 1$ which brings us to Step 4.

When $k > 1$, the procedure starts at a different positive integer, but the execution of steps is the same. Thus, it is necessary to explain what happens between Steps 3 and 4 to obtain a full appreciation of the method.

Example 1: Use Mathematical Induction to prove that $n^3 - n$ is divisible by 3, whenever n is a positive integer.

Solution: Let P_n be the proposition that " $n^3 - n$ is divisible by 3".

Basic Step: P_1 is true, since $1^3 - 1 = 0$ which is divisible by 3.

Inductive Step: Assume P_k is true, $k \in N$, $k > 1$: that is, $k^3 - k$ is divisible by 3.

We must show that P_{k+1} is true, if P_k is true. That is, $(k+1)^3 - (k+1)$ is divisible by 3.

$$\begin{aligned} \text{Now, } (k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= (k^3 - k) + 3(k^2 + k) \end{aligned}$$

Both terms are divisible by 3 since $(k^3 - k)$ is divisible by 3 by the assumption and $3(k^2 + k)$ is a multiple of 3. Hence, P_{k+1} is true whenever P_k is true.

Thus, $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Example 2:

Prove by Mathematical Induction that $7^n - 1$ is divisible by 3 for $\forall n \in N$.

Solution:

Let P_n be the statement " $7^n - 1$ is divisible by 3 for $\forall n \in N$ "

For $n = 1$, P is true, for $n = 2$, P is true. Hence $P_n = 3a$, for $a \in N$

Assuming that, P_k is true for $k = n$, where $n > 2$, $P_{k+1} = 3b$ for $b \in N$.

$$\begin{aligned} P_{k+1} - P_k &= 7^{k+1} - 1 - (7^k - 1) = 7^k (7 - 1) = 3(2 \times 7^k) \\ P_{k+1} &= 3(b - a). \end{aligned}$$

By the assumption for P_k for $k \in N$ then P_k is true $\forall n \in N$

Example 3: Prove by Mathematical Induction that the sum S_n of the first n odd positive integers is n^2 .

Solution: Let P_n be the proposition that the sum S_n of the first n odd positive integers is n^2 .

Basic Step: For $n = 1$ the first odd positive integer is 1, so $S_1 = 1$, that is: $S_1 = 1 = 1^2$, hence P_1 is true.

Inductive Step: Assume P_k is true, $k \in N$, $k > 1$. That is, $S_n = 1 + 3 + 5 + \dots$
 $+$

$$(2k - 1) = k^2.$$

$$\begin{aligned} \text{Now, } S_{k+1} &= 1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) \\ &= [1 + 3 + 5 + \dots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1), \text{ by the assumption,} \\ &= (k + 1)^2 \end{aligned}$$

Thus, P_{n+1} is true whenever P_k is true.

Since P_1 is true and $P_{k+1} \rightarrow P_{n+1}$ is true, the proposition P_n is true for all positive integers n .

Example 4: Prove by Mathematical Induction that $\sum_{r=1}^n r(r+1) = \frac{n}{3}(n+1)(n+2)$.

Solution: Let the statement $\sum_{r=1}^n r(r+1) = \frac{n}{3}(n+1)(n+2)$ be P_n .

For $n = 1$ and $n = 2$, P_n is true.

$$\text{Assuming that for } k = n, P_k = \sum_{r=1}^k r(r+1) = \frac{k}{3}(k+1)(k+2)$$

$$P_{k+1} = P_k + (k+1)(k+2)$$

$$\begin{aligned} \text{Adding the } (k+1)^{\text{th}} \text{ term gives } &= \frac{k}{3}(k+1)(k+2) + (k+1)(k+2) \\ &= \frac{k+1}{3}[(k+1)+1][(k+2)+1] \end{aligned}$$

By the assumption for $k = n$ then P_n is true $\forall n \in N$.

Functions (Injective, surjective, bijective, inverse)

Mathematical proof that a function is one-to-one (injective), onto (surjective) or (one-to-one and onto function) bijective should be introduced at this stage.

UNIT 1
MODULE 3: CALCULUS

Differential Equation

Example:

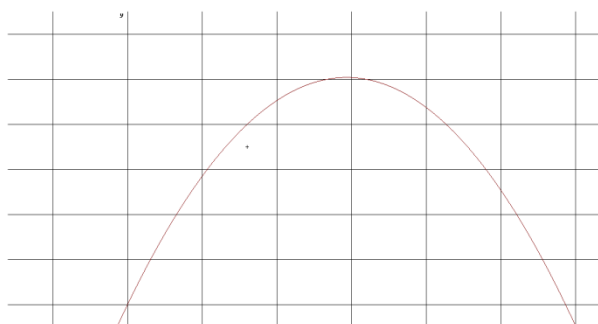
A particle moves along a path such that at time, t secs, it's velocity is given by $-3t^2 + 11t - 6 \text{ ms}^{-1}$.

- (a) Find the times at which the particle is momentarily at rest after being observed.
- (b) Sketch the graph of v on t .
- (c) At time 4 seconds after being observed the displacement, s metres, of the particle is 17.5 m. Find the displacement of the particle 7 seconds after being observed.

Solution:

- (a) The particle is at momentary rest when $v = 0$ i.e. when $(3t - 2)(t - 3) = 0$.
 $t = \frac{2}{3} \text{ sec}, 3 \text{ secs}$

- (b)



- (c) Given $\frac{ds}{dt} = 3t^2 - 11t + 6$ separating the variables gives $ds = 3t^2 - 11t + 6 dt$

Integrating both sides with the initial conditions gives $\int_{\frac{35}{2}}^s ds = \int_5^7 (3t^2 - 11t + 6) dt$

$$s - \frac{35}{2} = \left(t^3 - \frac{11}{2}t^2 + 6t \right)_5^7 \quad s = \frac{231}{2} \text{ metres}$$

UNIT 2

MODULE 1: COMPLEX NUMBERS AND CALCULUS II

Principal Argument of a Complex Number

The representation of the complex number $z = 1 + i$ on the Argand diagram may be used to introduce this topic. Encourage students to indicate and evaluate the argument of z . The students' answers should be displayed on the chalkboard.

Indicate that the location of z on the Argand diagram is unique, and therefore only one value of the argument is needed to position z . That argument is called the principal argument, $\arg z$, where:

$$-\pi < \text{principal argument} \leq \pi.$$

De Moivre's theorem to evaluate $\int e^{ax} \cos bx \, dx \left(\int e^{ax} \sin bx \, dx \right)$ by expressing for example

$$\operatorname{Re} \int e^{(a+bi)x} \, dx \left(\operatorname{Im} \int e^{(a+bi)x} \, dx \right)$$

$$\operatorname{Re} \int e^{(a+bi)x} \, dx = \operatorname{Re} \left(\frac{1}{a+bi} e^{(a+bi)x} + C \right) = \frac{a-bi}{a^2+b^2} e^{ax} (\cos bx + i \sin bx) + C$$

$$= \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2} e^{ax} (\cos bx + i \sin bx) + C$$

$$= \frac{e^{ax}}{a^2+b^2} (a \cos bx - b \sin bx) + C$$

The same approach can be used for $\int e^{ax} \sin bx \, dx = \operatorname{Im} \int e^{(a+bi)x} \, dx$. This is very useful in

reducing the repeated integration by parts of say $\int e^{2x} \cos 4x \, dx \left(\int e^{2x} \sin 4x \, dx \right)$

UNIT 2
MODULE 2: SEQUENCES, SERIES AND APPROXIMATIONS

The Binomial Theorem

Students may be motivated to do this topic by having successive expansions of $(a + x)^n$ and then investigating the coefficients obtained when expansions are carried out.

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + a^4$$

and so on.

By extracting the coefficients of each term made up of powers of a , x or a and x .

$$\begin{array}{ccccccc}
 & & & & 1 & & & & \\
 & & & & 1 & & 1 & & \\
 & & & 1 & & 2 & & 1 & \\
 & & 1 & & 3 & & 3 & & 1 \\
 1 & & 4 & & 6 & & 4 & & 1
 \end{array}$$

Encourage students to use the emerging pattern to generate further expansions of $(a + x)^n$. This can be done by generating the coefficients from Pascal's Triangle and then investigating other patterns. For example, by looking at the powers of a and x (powers of x increase from 0 to n , while powers of a decrease from n to 0; powers of a and x add up to n).

In discussing the need to find a more efficient method of doing the expansions, the Binomial Theorem may be introduced. However, this can only be done after the students are exposed to principles of counting, with particular reference to the process of selecting. In so doing, teachers will need to guide students through appropriate examples involving the selection of r objects, say, from a group of n unlike objects. This activity can lead to defining ${}^nC_r$ as the number of ways of selecting r objects from a group of n unlike objects.

In teaching this principle, enough examples should be presented before ${}^nC_r = \frac{n!}{(n-r)!r!}$ formula is developed.

The binomial theorem may then be established by using the expansion of $(1 + x)^n$ as a starting point. A suggested approach is given below:

Consider $(1 + x)^n$.

To expand, the student is expected to multiply $(1 + x)$ by itself n times, that is, $(1 + x)^n = (1 + x)(1 + x)(1 + x) \dots (1 + x)$.

The result of the expansion is found as given below:

The constant term is obtained by multiplying all the 1's. The result is therefore 1.

The term in x is obtained by multiplying $(n - 1)$ 1's and one x . This x , however, may be chosen from any of the n brackets. That is, we need to choose one x out of n different brackets. This can be done in ${}^nC_1$ ways. Hence, the coefficient of x is ${}^nC_1$.

Similarly, the term in x^2 may be obtained by choosing two x 's and $(n - 2)$ 1's. The x 's may be chosen from any two of the n brackets. This can be done in ${}^nC_2$ ways. The coefficient of x^2 is therefore ${}^nC_2$.

This process continues and the expansion is obtained:

$$(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots + x^n$$

This is known as the binomial theorem. The theorem may be written as

$$(1 + x)^n = \sum_{r=0}^n {}^nC_r x^r$$

The generalisation of this could be done as a class activity where students are asked to show that:

$$(a + b)^n = a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \dots + b^n$$

This is the binomial expansion of $(a + b)^n$ for positive integral values of n . The expansion terminates after $(n + 1)$ terms.

For $n \in \mathbb{Q}$, it should be demonstrated to the student that the notation

$$n! = n(n - 1)(n - 2)(n - 3) \dots \times 1$$

Is applicable to the expansion of

$$(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \frac{n(n - 1)(n - 2)}{3!}x^3 + \frac{n(n - 1)\dots(n - r + 1)}{r!} + \dots$$

where $n \in \mathbb{Q}$.

Existence of Roots

Introduce the existence of the root of a continuous function $f(x)$ between given values a and b as an application of the Intermediate Value Theorem.

Emphasis should be placed on the fact that:

- (a) f must be continuous between a and b ;
- (b) The product of $f(a)$ and $f(b)$ is less than zero, that is, $f(a)$ and $f(b)$ must have opposite signs.

Linear Interpolation

Engage students in the activity below.

Given the points (x_0, y_0) and (x_1, y_1) on a continuous curve $y = f(x)$, students can establish that for $f(x_0)$ and $f(x_1)$ with opposite signs and that f is continuous, then

$x_0 < x < x_1$, for the equation $f(x) = 0$. If $|f(x_0)| < |f(x_1)|$ say, students can be introduced to the concept of similar triangles to find successive approximations, holding $f(x_1)$ constant.

This intuitive approach is formalised in **linear interpolation**, where the two points (x_0, y_0) and (x_1, y_1) can be joined by a straight line and the x -value of the point on this line is calculated. A first approximation for x can be found using

$$\frac{x_0}{|f(x_0)|} = \frac{x_1}{|f(x_1)|} .$$

Successive approximations can be found with this approach until the same answer to the desired degree of accuracy is obtained.

UNIT 2

MODULE 3: COUNTING, MATRICES AND DIFFERENTIAL EQUATIONS

Counting

Allow students to consider the three scenarios given below.

- (a) Throw two dice. Find the probability that the sum of the dots on the uppermost faces of the dice is 6.
- (b) An insurance salesman visits a household. What is the probability that he will be successful in selling a policy?
- (c) A hurricane is situated 500km east of Barbados. What is the probability that it will hit the island?

These three scenarios are very different for the calculation of probability. In (a), the probability is calculated as the number of successful outcomes divided by the total possible number of outcomes. In this classical approach, the probability assignments are based on equally likely outcomes and the entire sample space is known from the start.

The situation in (b) is no longer as well determined as in 'a'. It is necessary to obtain historical data for the salesman in question and estimate the required probability by dividing the number of successful sales by the total number of households visited. This frequency approach still relies on the existence of data and its applications are more realistic than those of the classical methodology.

For (c) it is very unclear that a probability can be assigned. Historical data is most likely unavailable or insufficient for the frequency approach. The statistician might have to revert to informed educated guesses. This is quite permissible and reflects the analyst's prior opinion. This approach lends itself to a Bayesian methodology.

One should note that the rules and results of probability theory remain exactly the same regardless of the method used to estimate the probability of events.

Western Zone Office

15 August 2022

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PURE MATHEMATICS

Specimen Papers and Mark Schemes/Keys

Specimen Papers, Mark Schemes and Keys:

Unit 1 Paper 01
Unit 1 Paper 02
Unit 1 Paper 032
Unit 2 Paper 01
Unit 2 Paper 02
Unit 2 Paper 032

SPECIMEN 2022



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ALGEBRA, GEOMETRY AND CALCULUS

PURE MATHEMATICS

UNIT 1 – Paper 01

90 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This test consists of 45 items. You will have 90 minutes to answer them.
2. Each item in this test has four suggested answers lettered (A), (B), (C), (D). Read each item you are about to answer and decide which choice is best.
3. Look at the sample item below.

Sample Item

The lines $2y - 3x - 13 = 0$ and $y + x + 1 = 0$
intersect at the point

- (A) $(-3, -2)$
- (B) $(-3, 2)$
- (C) $(3, -2)$
- (D) $(3, 2)$

Sample Answer



The best answer to this item is “ $(-3, 2)$ ”, so answer space (B) has been shaded.

4. You may do any rough work in this booklet.
5. The use of silent, non-programmable calculators is allowed

Examination Materials Permitted

A list of mathematical formulae and tables (provided) – **Revised 2010**

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

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1. Let p , q and r be the propositions
 p : Students have a driving licence,
 q : Students have a passport,
 r : Students have an identification card.

The compound proposition, Students have driving licence or identification card (but not both) together with a passport is expressed as

- (A) $((p \wedge r) \vee \sim(p \wedge r)) \wedge q$
 (B) $((p \vee r) \vee \sim(p \wedge r)) \vee q$
 (C) $((p \vee r) \vee \sim(p \vee r)) \wedge q$
 (D) $((p \vee r) \wedge \sim(p \wedge r)) \wedge q$

2. The compound proposition $p \wedge q$ is true can be illustrated by the truth table

(A)

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

(B)

p	q	$p \wedge q$
0	0	1
0	1	0
1	0	0
1	1	1

(C)

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	1
1	1	1

(D)

p	q	$p \wedge q$
0	0	1
0	1	0
1	0	1
1	1	0

3. The contrapositive for the conditional proposition $p \rightarrow q$ is

- (A) $q \rightarrow p$
 (B) $\sim p \rightarrow q$
 (C) $\sim q \rightarrow \sim p$
 (D) $p \rightarrow \sim q$

4. The proposition $q \rightarrow p$ is logically equivalent to

- (A) $\sim p \wedge \sim q$
 (B) $p \vee \sim q$
 (C) $\sim q \wedge p$
 (D) $q \wedge \sim p$

5. If a remainder of 7 is obtained when $x^3 - 3x + k$ is divided by $x - 3$, then k equals

- (A) -11
 (B) -1
 (C) 1
 (D) 11

6. Give that $x = 3^y$, $y > 0$ then $\log_x 3$ is equal to

- (A) y
 (B) $3y$
 (C) $\frac{1}{y}$
 (D) $\frac{3}{y}$

GO ON TO THE NEXT PAGE

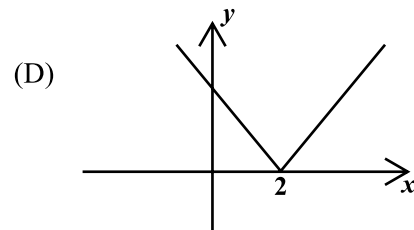
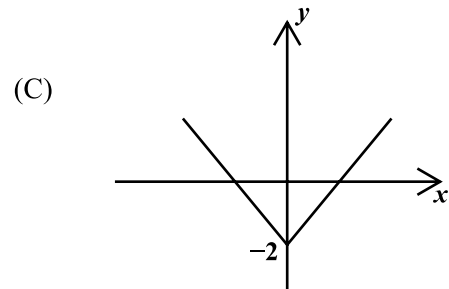
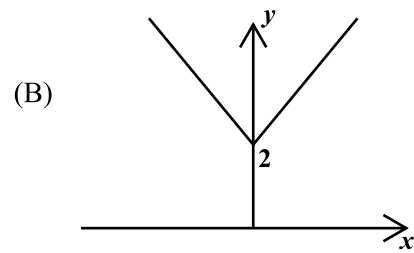
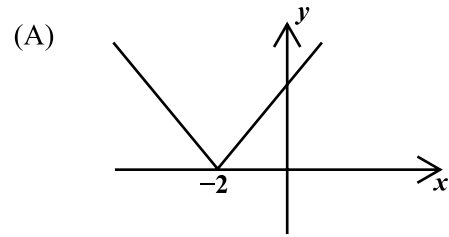
7. Given that $f(x) = 2 - e^{2x}$, the inverse function, $f^{-1}(x)$, for $x < 2$ is

- (A) $\ln(2 - x)$
- (B) $\ln(2 - 2x)$
- (C) $2 \ln(2 - x)$
- (D) $\frac{1}{2} \ln(2 - x)$

8. Given that $fg(x) = x$, where $g(x) = \frac{2x+1}{3}$, $f(x) =$

- (A) $\frac{3x-1}{2}$
- (B) $\frac{3}{2x+1}$
- (C) $\frac{2}{3x+1}$
- (D) $\frac{3x}{2x+1}$

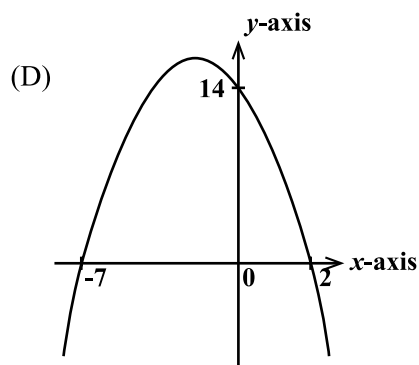
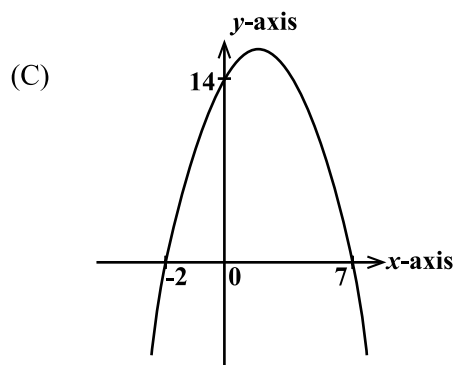
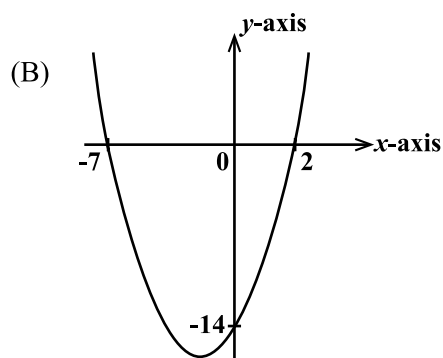
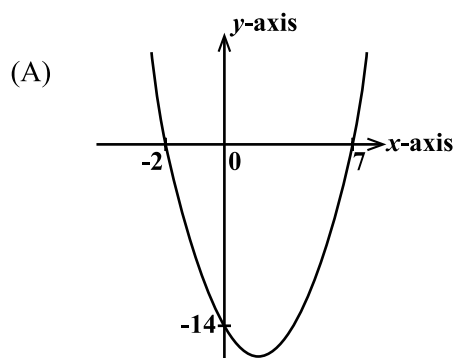
9. If $f(x) = |x|$, which of the diagrams below represents the graph of $y = f(x) + 2$?



10. If $3^{2x+1} - 4(3^x) + 1 = 0$ then which of the statements below is true?
- I. $x = -1$
II. $x = 1$
III. $x = 0$
- (A) I only
(B) I and II only
(C) I and III only
(D) II and III only
11. $\frac{2\sqrt{3} + 3\sqrt{2}}{\sqrt{3} + \sqrt{2}}$ can be simplified correctly to
- (A) $\sqrt{6}$
(B) $2\sqrt{5}$
(C) $12 + 5\sqrt{6}$
(D) $\frac{12 + 5\sqrt{6}}{5}$
12. The cubic equation $2x^3 + x^2 - 22x + 24 = 0$ has roots α, β and γ . The value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is
- (A) $-\frac{1}{12}$
(B) $-\frac{1}{11}$
(C) $-\frac{1}{2}$
(D) $\frac{11}{12}$
13. The range (s) of values of x for which $\frac{3x+2}{x-1} > 0$ are
- (A) $x > -\frac{2}{3}, x > 1$
(B) $-\frac{2}{3} < x < 1$
(C) $x < \frac{2}{3}, x > 1$
(D) $x < -\frac{2}{3}, x > 1$

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14. Which one of the graphs below best represents the equation $y = x^2 - 5x - 14$?



15. The values of x for which $|x + 5| > 3$ are

- (A) $x < -8, x < -2$
 (B) $x > 0, x < 1$
 (C) $x > -2, x < -8$
 (D) $x > -2, x > 8$

16.
$$\frac{1 + \cot^2 \theta}{\sec \theta \operatorname{cosec} \theta} =$$

- (A) $\tan \theta$
 (B) $\cos \theta$
 (C) $\cot \theta$
 (D) $\operatorname{cosec} \theta$

17. The general solution of the equation $\cos 2\theta = 1$ is

- (A) $n\pi + \frac{\pi}{4}$
 (B) $n\pi$
 (C) $n\pi + \frac{\pi}{2}$
 (D) $\frac{(2n+1)\pi}{4}$

18. $\cos \theta + 3 \sin \theta = 2$ can be expressed as

- (A) $4 \cos \left(\theta - \tan^{-1} \left(\frac{1}{3} \right) \right) = 2$
 (B) $2 \cos \left(\theta + \tan^{-1} (3) \right) = 2$
 (C) $\sqrt{10} \cos \left(\theta - \tan^{-1} (3) \right) = 2$
 (D) $\sqrt{10} \cos \left(\theta + \tan^{-1} \left(\frac{1}{3} \right) \right) = 2$

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19. If $\cos A = \frac{3}{5}$ and A is acute, then $\sin 2A$ is equal to
- (A) $\frac{6}{25}$
(B) $\frac{8}{25}$
(C) $\frac{12}{25}$
(D) $\frac{24}{25}$
20. The maximum value of $\frac{1}{2\cos\left(\theta + \frac{\pi}{4}\right)}$ is
- (A) -1
(B) 0
(C) $\frac{1}{2}$
(D) 2
21. A curve C_1 is given by the equation $y = x^2 + 1$, and a curve C_2 is given by the equation $\frac{16}{x^2} + 1, x \in \mathbb{R}, x > 0$. The value of x for which $C_1 = C_2$ is
- (A) -4
(B) 2
(C) -2
(D) 4
22. The tangent to the circle, C , with equation $x^2 + y^2 + 4x - 10y - 5 = 0$ at the point $P(3, 2)$ has equation
- (A) $3x + 5y - 19 = 0$
(B) $5x + 3y + 19 = 0$
(C) $3x - 5y + 19 = 0$
(D) $5x - 3y - 9 = 0$
23. The centre of the circle $(x - 3)^2 + (y + 2)^2 = 25$ is
- (A) $(3, -2)$
(B) $(2, -3)$
(C) $(-3, 2)$
(D) $(-2, 3)$
24. The Cartesian equation of the curve C given by the parametric equations $x = 3 \sin \theta - 2$, $y = 4 \cos \theta + 3$ is
- (A) $x^2 + y^2 = 309$
(B) $9x^2 + 16y^2 = 13$
(C) $(x - 3)^2 + 4(y - 4)^2 = 36$
(D) $16(x + 2)^2 + 9(y - 3)^2 = 144$
25. Relative to a fixed origin, O , the position vector of A is $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and the position vector of B is $\overrightarrow{OB} = 9\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$. $|\overrightarrow{AB}|$ is
- (A) 1 unit
(B) 7 units
(C) $3\sqrt{21}$ units
(D) 49 units

26. Relative to a fixed origin, O , the point A has position vector $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$, and point B has position vector $(-5\mathbf{i} + 9\mathbf{j} - 5\mathbf{k})$. The line, l , passes through the points A and B . A vector equation for the line l is given by

- (A) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-7\mathbf{i} + 6\mathbf{j} - \mathbf{k})$
(B) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-5\mathbf{i} + 9\mathbf{j} - 5\mathbf{k})$
(C) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-3\mathbf{i} + 12\mathbf{j} - 9\mathbf{k})$
(D) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(-10\mathbf{i} + 27\mathbf{j} + 20\mathbf{k})$

27. Relative to a fixed origin, O , the line, P_1 , has position vector $\begin{pmatrix} 8 \\ 13 \\ -12 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$,

and the line, P_2 , has position vector

$$\begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix} \text{ where } \lambda \text{ and } \mu \text{ are}$$

scalars.

The cosine of the acute angle between P_1 and P_2 is given by

- (A) $\cos \theta = -\frac{2}{3}$
(B) $\cos \theta = \frac{2}{3}$
(C) $\cos \theta = \frac{8}{27}$
(D) $\cos \theta = \frac{2}{27}$

28. Relative to a fixed origin, O , the point A has position vector $(10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k})$, and the point B has position vector $(5\mathbf{i} + 9\mathbf{j} + 6\mathbf{k})$. Given that a vector $\mathbf{v}$ is of magnitude $3\sqrt{6}$ units in the direction of $\overrightarrow{AB}$, then $\mathbf{v} =$

- (A) $3\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$
(B) $-3\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$
(C) $-3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$
(D) $3\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

29. The value that θ , $0 \leq \theta \leq \pi$, which satisfies the equation $2\cos^2\theta + 3\cos\theta - 2 = 0$ is

- (A) $\frac{\pi}{6}$
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{2}$

30. If the length of the vector $x = 5\mathbf{i} - (k - 2)\mathbf{j}$ is $\sqrt{34}$ and k is real, then $k =$

- I. 5
II. -1
III. -5

- (A) III only
(B) I and II only
(C) II and III only
(D) I, II and III

31. $\lim_{x \rightarrow 3} \frac{2x^2 - 5x - 3}{x^2 - 2x - 3}$ is
- (A) 0
- (B) 1
- (C) $\frac{7}{4}$
- (D) ∞
32. $\lim_{x \rightarrow 0} \frac{\sin x}{\frac{x}{2}}$ is
- (A) 0
- (B) $\frac{1}{2}$
- (C) 1
- (D) 2
33. Given that $\lim_{x \rightarrow -1} \{3f(x) + 2\} = 11$, where $f(x)$ is real and continuous, the $\lim_{x \rightarrow -1} \{2f(x) + 5x\}$ is
- (A) -11
- (B) 1
- (C) 4
- (D) 13
34. Given that $f(x) = (6x + 4) \sin x$, then $f'(x)$ is
- (A) $6 \cos x$
- (B) $2(3x + 2) \cos x + 6 \sin x$
- (C) $6x \cos x + 6 \sin x$
- (D) $3 + 2 \sin x + (3x + 2) \cos x$
35. The derivative by first principles of the function $f(x) = \frac{1}{x^2}$ is given by
- (A) $\lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$
- (B) $\lim_{h \rightarrow 0} \frac{\frac{1}{(x^2+h)} - \frac{1}{x^2}}{h}$
- (C) $\lim_{h \rightarrow 0} \frac{\frac{1}{x^2} - \frac{1}{(x^2+h)}}{h}$
- (D) $\lim_{h \rightarrow 0} \frac{1}{(x+h)^2} - \frac{1}{x^2}$
36. If $y = \cos x - \sin x$, then $\frac{d^2 y}{dx^2}$ is
- (A) $\cos x + \sin x$
- (B) $\cos x - \sin x$
- (C) $-\cos x + \sin x$
- (D) $-\cos x - \sin x$

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37. At $x = 2$, the function $2x^3 - 6x^2 + 5x$
- (A) is decreasing
 - (B) is increasing
 - (C) has a minimum value
 - (D) has a maximum value
38. Given $f(x) = (x - 2)(x^3 + 5)$, $f'(x)$ is?
- (A) $3x^2$
 - (B) $-2(3x^2 + 5)$
 - (C) $4x^3 - 6x^2 + 5$
 - (D) $x^4 - 2x^3 + 5x - 10$
39. The curve, C , with equation $y = x^3 - 6x^2 + 9x$ has stationary points at $P(3, 0)$ and $Q(1, 4)$. The nature of these stationary points is
- (A) $(3, 0)_{\max} (1, 4)_{\max}$
 - (B) $(3, 0)_{\text{infl}} (1, 4)_{\min}$
 - (C) $(3, 0)_{\text{infl}} (1, 4)_{\max}$
 - (D) $(3, 0)_{\min} (1, 4)_{\max}$
40. Given that $\frac{d}{dx} \frac{2x-1}{3x+2} = \frac{7}{(3x+2)^2}$ then
- $$\int_1^2 \frac{21}{(3x+2)^2} dx =$$
- (A) $\left. \frac{3(7)}{(3x+2)^2} \right|_1^2$
 - (B) $\left. \frac{2x-1}{(3x+2)} \right|_1^2$
 - (C) $\left. \frac{3(2x-1)}{(3x+2)} \right|_1^2$
 - (D) $\left. \frac{-21}{(3x+2)} \right|_1^2$
41. The area of the finite region, R , enclosed by the curve $y = x - \frac{1}{\sqrt{x}}$, the lines $x = 1$ and $x = 4$ is
- (A) $\frac{9}{2}$
 - (B) $\frac{11}{2}$
 - (C) $\frac{27}{4}$
 - (D) $\frac{19}{2}$
42. The volume (in units³) generated when the region bounded by the graphs of $y^2 = x + 3$, $x = 0$ and $x = 3$ is rotated through 2π radians about the x -axis is
- (A) 63
 - (B) $\frac{27}{2}$
 - (C) 63π
 - (D) $\frac{27}{2}\pi$

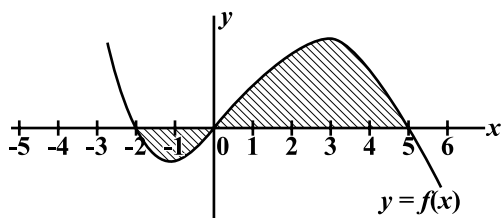
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43. Given that $\int_1^3 f(x) dx = 8$, then

$$\int_1^3 [2f(x) - 5] dx =$$

- (A) 6
(B) 11
(C) 13
(D) 21

44. The total shaded area in the diagram below is given by



- (A) $\int_{-2}^5 f(x) dx$
(B) $-\int_{-2}^0 f(x) dx + \int_0^5 f(x) dx$
(C) $\int_{-5}^{-2} f(x) dx$
(D) $\int_{-2}^0 f(x) dx + \int_0^5 f(x) dx$

45. Given that $\int_{-2}^0 f(x) dx = \frac{16}{3}$ and water is pumped into a large tank at a rate that is proportional to its volume, V , $\int_{-2}^2 f(x) dx = \frac{32}{3}$ where $f(x)$ is a real continuous function in the closed interval $[-2, 2]$, then $\int_0^2 f(x) dx =$

- (A) $\frac{16}{3}$
(B) 16
(C) $\frac{64}{3}$
(D) 32

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

***DO NOT
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CANDIDATE'S RECEIPT

INSTRUCTIONS TO CANDIDATE:

1. Fill in all the information requested clearly in capital letters.

TEST CODE:

0	2	1	3	4	0	1	0
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SUBJECT: PURE MATHEMATICS – UNIT 1 – Paper 01

PROFICIENCY: ADVANCED

REGISTRATION NUMBER:

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FULL NAME: _____
(BLOCK LETTERS)

Signature: _____

Date: _____

2. Ensure that this slip is detached by the Supervisor or Invigilator and given to you when you hand in this booklet.
3. Keep it in a safe place until you have received your results.

INSTRUCTION TO SUPERVISOR/INVIGILATOR:

Sign the declaration below, detach this slip and hand it to the candidate as his/her receipt for this booklet collected by you.

I hereby acknowledge receipt of the candidate's booklet for the examination stated above.

Signature: _____
Supervisor/Invigilator

Date: _____

UNIT 1

PAPER 01

Item	Key	Specific Obj.
1	D	1.1, 1.2
2	A	1.1, 1.2
3	C	1.3
4	B	1.4
5	A	3.2
6	C	5.1
7	D	5.6
8	A	4.2
9	B	6.4
10	C	5.6
11	A	2.3
12	D	7.1
13	D	4.1
14	A	4.5
15	C	6.5

Item	Key	Specific Obj.
16	C	1.1
17	B	1.8
18	C	1.10
19	D	1.5
20	C	1.12
21	B	2.4
22	D	2.2
23	A	2.1
24	D	2.5
25	B	3.3
26	A	3.7
27	A	3.6
28	B	3.4
29	C	1.11
30	B	3.3

Item	Key	Specific Obj.
31	D	1.4
32	D	1.5
33	B	1.4
34	B	2.5
35	A	2.3
36	C	2.10
37	B	2.8
38	C	2.5
39	D	2.12
40	C	3.1
41	B	3.9
42	D	3.10
43	A	3.8
44	B	3.9
45	A	3.8

SPECIMEN 2022



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C A R I B B E A N E X A M I N A T I O N S C O U N C I L

C A R I B B E A N A D V A N C E D P R O F I C I E N C Y E X A M I N A T I O N[®]

PURE MATHEMATICS

UNIT 1

ALGEBRA, GEOMETRY AND CALCULUS

SPECIMEN PAPER

PAPER 02

2 hours 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of TWO questions.
3. Answer ALL questions from the THREE sections.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet.
Remember to draw a line through your original answer.
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2022**

Electronic calculator

Ruler and graph paper

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02134020/SPEC/CAPE 2022

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Sequential Bar Code

SECTION A

Module 1

Answer BOTH questions.

1. (a) (i) Let p and q be any two propositions. Complete the truth table below.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T					
T	F					
F	T					
F	F					

[4 marks]

- (ii) Hence, state whether the statements $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent. Justify your response.

[2 marks]

- (b) Solve the equation $|x^2 - x - 4| = 2$ for all real values of x .

[6 marks]

GO ON TO THE NEXT PAGE

- (c) (i) Determine the value(s) of the constants p and q such that $x - 2$ is a common factor of $x^3 - x^2 - 2px + 3q$ and $qx^3 - px^2 + x + 2$.

[10 marks]

- (ii) Hence, determine all roots of the equation $x^3 - x^2 - 2px + 3q$.

[3 marks]

Total 25 marks

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2. (a) A function f is defined as $f(x) = \log_3(x + 3)$.

(i) Determine

a) the domain of the function

b) the range of the function

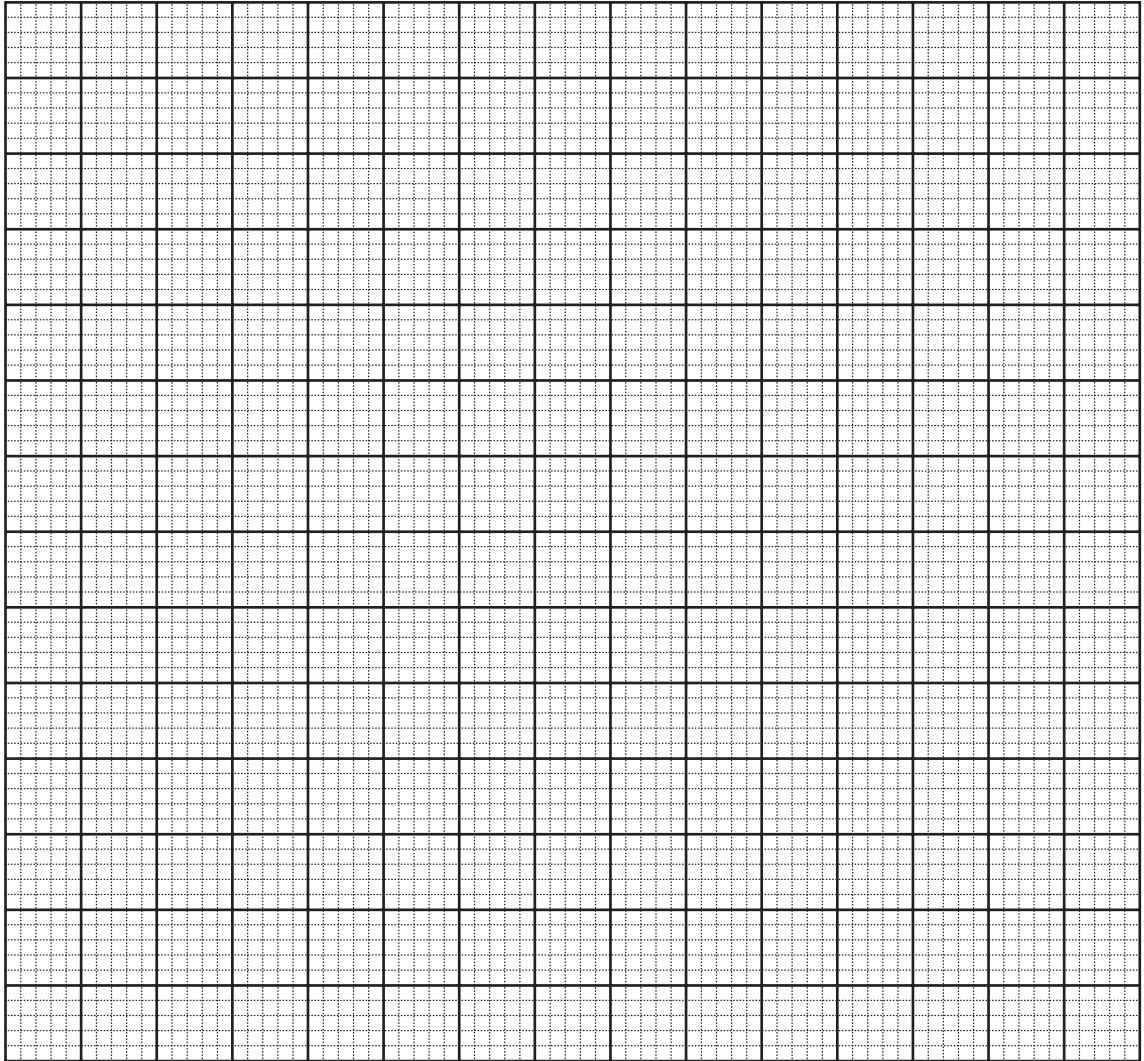
[2 marks]

(ii) Determine the inverse function $f^{-1}(x)$.

[3 marks]

GO ON TO THE NEXT PAGE

- (iii) On the grid provided below, sketch the graph of $f(x)$ and $f^{-1}(x)$, clearly indicating the intercepts.



[7 marks]

- (b) Prove by mathematical induction that $\sum_r^n = 2 \frac{1}{(r-1)r} = 1 - \frac{1}{n}$ is true for all values $n \geq 2$.

[6 marks]

- (c) The cubic equation $2x^3 - 4x^2 + 5x - 3 = 0$ has roots α , β and γ .

State the value of:

(i) $\sum \alpha$

[1 mark]

(ii) $\sum \alpha\beta$

[1 mark]

(iii) $\alpha\beta\gamma$

[1 mark]

(iv) Hence, determine the value of $\frac{1}{\alpha^3\beta^3} + \frac{1}{\beta^3\gamma^3} + \frac{1}{\alpha^3\gamma^3}$

[4 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION B

Module 2

Answer BOTH questions.

3. (a) Prove the identity $\frac{\sin 3\theta + \sin 5\theta}{\cos 3\theta + \cos 5\theta} = \tan 4\theta$.

[4 marks]

- (b) (i) Show that the equation $2 \sin \theta - \cot \theta = 0$ may be written as $2 \cos^2 \theta + \cos \theta - 2 = 0$

[3 marks]

- (ii) Hence, determine the general solution for θ .

[4 marks]

GO ON TO THE NEXT PAGE

- (c) (i) Express $\sqrt{3} \sin \theta + \cos \theta$ in the form $R \sin(\theta + a)$ where $R > 0$ and $0 < a < \frac{\pi}{2}$.

[3 marks]

- (ii) Hence, determine the maximum value of $f(\theta) = \frac{1}{\sqrt{3} \sin \theta + \cos \theta + 3}$.

[5 marks]

- (d) Determine whether the lines $l_1 : r = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$ and $l_2 : r = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ intersect.

[6 marks]

Total 25 marks

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4. (a) Prove that the parametric equations $x = 3 \sin \theta + 5$, $y = 2 \cos \theta + 6$ represent the equation of an ellipse.

[3 marks]

- (b) (i) B is the point $(3, 4)$. A point P moves so that the fixed distance from point B is $\sqrt{5}$ units. Show that the locus of P is a circle, C , with centre $(3, 4)$.

[3 marks]

- (ii) Determine the equation of the tangent and normal to the circle, C , at the point $(5, 5)$.

[4 marks]

- (iii) Determine the coordinates of the points of intersection of the circle C and the line $3y = -x + 10$.

[4 marks]

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(c) Let $A = (-1, 1, -1)$, $B = (-2, -1, -4)$ and $C = (-4, 2, -5)$.

(i) Show that the vector $\begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix}$ is perpendicular to the plane A, B and C .

[5 marks]

(ii) Hence, determine the Cartesian equation of the plane, π_1 , through A, B and C .

[3 marks]

(iii) If the angle between two planes is the angle between their normal vectors, determine the angle between π_1 and $\pi_2: 2x - y + 2z = 3$.

[3 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION C

Module 3

Answer BOTH questions.

5. (a) (i) The function $f(x)$ is defined as $f(x) = \left(\frac{2x^2 - 4x + 5}{5x} + \frac{5x - 4}{x} \right)$.

Determine $\lim_{x \rightarrow 1} f(x)$

[3 marks]

- (ii) Differentiate $f(x) = x^3 - 6x$ from first principles.

[5 marks]

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- (b) The function $g(x)$ takes the form $ax^4 + bx^2 + c$. Given that $g''(x) = 36x^2 - 10$, $g(1) = 4$ and $g'(0) = 0$, determine the values of a , b and c .

[7 marks]

- (c) A spherical balloon is being inflated by an air pump. The rate of increase of the radius of the balloon is 1.5 cm/s.
- (i) Determine the rate at which the volume of the balloon is increasing when the radius is 5 cm, giving your answer in terms of π .

[5 marks]

GO ON TO THE NEXT PAGE

- (ii) Given that the maximum volume of the balloon is 2500 cm^3 , determine
- a) the time it takes to completely inflate the balloon.

[3 marks]

- b) the radius of the balloon when it is fully inflated.
Use $\pi = 3.14$

[The volume, V , of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$]

[2 marks]

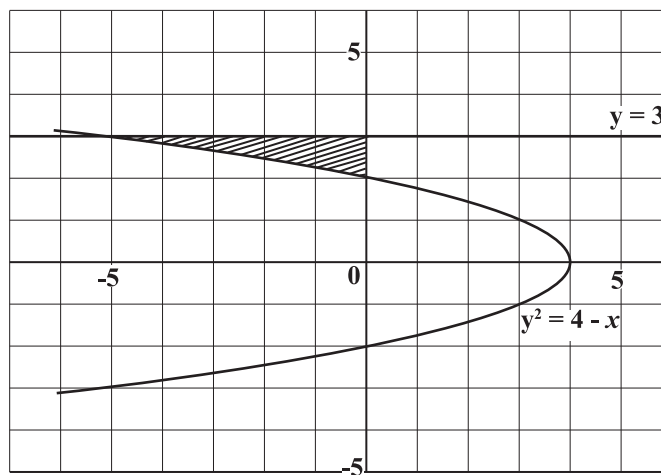
Total 25 marks

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6. (a) Differentiate $y = (2x^3 + 3) \cos(x^2)$ with respect to x .

[4 marks]

- (b) The diagram below shows the region bounded by the curve $y^2 = 4 - x$, the line $y = 3$ and the y -axis.



- (i) Calculate the area of the shaded region.

[6 marks]

GO ON TO THE NEXT PAGE

- (ii) Calculate the volume of the solid generated when the shaded region is rotated completely about the line $y = 3$, giving your answer in terms of π .

[5 marks]

- (c) The function $f(x)$ is defined by $f(x) = x^3 - \frac{7}{2}x^2 + 2x$.

Determine the coordinates and the nature of the stationary points of $f(x)$.

[10 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

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Question No.

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INSTRUCTIONS TO CANDIDATE:

1. Fill in all the information requested clearly in capital letters.

TEST CODE:

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SUBJECT: PURE MATHEMATICS – UNIT 1 – Paper 02

PROFICIENCY: ADVANCED

REGISTRATION NUMBER:

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2. Ensure that this slip is detached by the Supervisor or Invigilator and given to you when you hand in this booklet.
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Supervisor/Invigilator

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C A R I B B E A N E X A M I N A T I O N S C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATIONS®

PURE MATHEMATICS

UNIT 1 – Paper 02

KEY AND MARK SCHEME

MAY/JUNE 2022

SPECIMEN PAPER

PURE MATHEMATICS
UNIT 1 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total																																			
1.	(a) (i)	<table><tr><th>p</th><th>q</th><th>$p \wedge q$</th><th>$\sim(p \wedge q)$</th><th>$\sim p$</th><th>$\sim q$</th><th>$\sim p \vee \sim q$</th></tr><tr><td>T</td><td>T</td><td>T</td><td>F</td><td>F</td><td>F</td><td>F</td></tr><tr><td>T</td><td>F</td><td>F</td><td>T</td><td>F</td><td>T</td><td>T</td></tr><tr><td>F</td><td>T</td><td>F</td><td>T</td><td>T</td><td>F</td><td>T</td></tr><tr><td>F</td><td>F</td><td>F</td><td>T</td><td>T</td><td>T</td><td>T</td></tr></table> (1) (1) (1) (1) either column	p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	T	T	T	F	F	F	F	T	F	F	T	F	T	T	F	T	F	T	T	F	T	F	F	F	T	T	T	T	4
p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$																																
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	(ii)	Since $\sim(p \wedge q)$ and $\sim p \vee \sim q$ have the same values in the truth table, they are logically equivalent. [2 marks]	2																																			
	(b)	$ x^2 - x - 4 = 2$ Either $x^2 - x - 4 = 2$ OR $x^2 - x - 4 = -2$ [2 marks] $x^2 - x - 4 = 2$ $x^2 - x - 6 = 0$ $(x - 3)(x + 2) = 0$ $x = -2, 3$ [2 marks] OR $x^2 - x - 4 = -2$ $x^2 - x - 6 = -2$ $x^2 - x - 2 = 0$ $(x + 1)(x - 2) = 0$ $x = -1, 2$ [2 marks]	6																																			

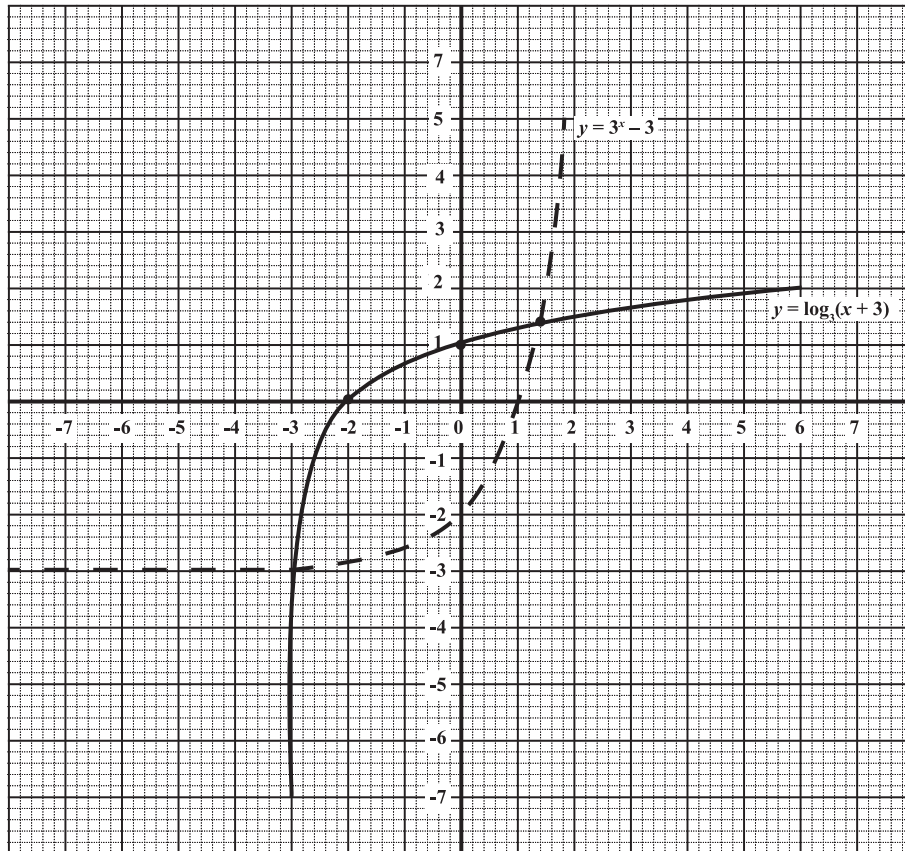
PURE MATHEMATICS
UNIT 1 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(c) (i)	If $x-2$ is a factor of $x^3 - x^2 - 2px + 3q$ then $f(2) = 0$ [1 mark] <i>i.e.</i> $4p - 3q = 4$ Equation 1 [2 marks] If $x-2$ is a factor of $qx^3 - px^2 + x + 2$ then $g(2) = 0$ [1 mark] <i>i.e.</i> $4p - 8q = 4$ Equation 2 [2 marks] $4p - 3q = 4$ $4p - 8q = 4$ Recognizing that equations can be solved simultaneously [1 mark] SOI Attempt to solve simultaneous equations [1 mark] Solution $p = 1, q = 0$ [2 marks]	10
	(ii)	$x-2 \overline{)x^3 - 2x = x^2 + x}$ $\therefore x^3 - x^2 - 2x = x(x+1)(x-2)$ $(x+1)(x-2) = 0$ $x = -1, 0, 2$ [3 marks]	3
		TOTAL	25
		Specific Objectives: 1.1, 1., 1.4, 3.1, 3.3, 6.1, 6.2, 6.5	

PURE MATHEMATICS
UNIT 1 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
2.	(a) (i)	Domain: $x > -3$ or $(-3, \infty)$ [1 mark] Range: $(-\infty, \infty)$ [1 mark]	2
	(ii)	$f(x) = \log_3(x+3)$ $y = \log_3(x+3)$ $x = \log_3(y+3)$ $3^x = y+3$ $y = 3^x - 3 = f^{-1}(x)$ [3 marks]	3
	(iii)	$y = \log_3(x+3)$ When $x = 0$, $y = \log_3 3 = 1$ y -intercept $(0, 1)$ [1 mark] When $y = 0$, $0 = \log_3(x+3)$ $1 = x+3$ $x = -2$ x -intercept $(-2, 0)$ [1 mark] $y = 3^x - 3$ When $x = 0$ $y = 3^0 - 3 = -2$ y -intercept $(0, -2)$ [1 mark] When $y = 0$ $0 = 3^x - 3$ $3 = 3^x$ $x = 1$ x -intercept $(1, 0)$ [1 mark]	7

PURE MATHEMATICS
UNIT 1 – Paper 02
KEY AND MARK SCHEME



Graph showing $y = \log_3(x + 3)$

[1 mark]

Graph showing $y = 3^x - 3$

[1 mark]

Showing all 4 intercepts

[1 mark]

PURE MATHEMATICS
UNIT 1 – Paper 02
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	Question	Solutions	Total
	(b)	Verifying statement true for $n = 1$ [1 mark] Assume statement true for $n = k$ [1 mark] $\sum_{r=2}^n \frac{1}{(r-1)r} = 1 - \frac{1}{k}$ $\sum_{r=2}^{k+1} \frac{1}{(r-1)r} = \sum_{r=2}^k \frac{1}{(r-1)r} + (k+1)^{th} \text{ term} [2 \text{ marks}]$ $\sum_{r=2}^{k+1} \frac{1}{(r-1)r} = \left(1 - \frac{1}{k}\right) + \frac{1}{k(k+1)} [1 \text{ mark}]$ Proving the statement true for $n = k + 1$ $\sum_{r=2}^{k+1} \frac{1}{(r-1)r} = 1 - \frac{1}{k+1}$ Hence by the Principle of Mathematical Induction, the statement is true for all values $n \geq 2$. [1 mark]	6
	(c) (i)	$2x^3 - 4x^2 + 5x - 3 = 0$ $(\div 2)x^3 - 2x^2 + \frac{5}{2}x - \frac{3}{2} = 0 [1 \text{ mark}]$ $\sum \alpha = -(-2) = 2$	1
	(ii)	$\sum \alpha\beta = \frac{5}{2} [1 \text{ mark}]$	1
	(iii)	$\alpha\beta\gamma = \frac{3}{2} [1 \text{ mark}]$	1

PURE MATHEMATICS
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	Question	Solutions	Total
	(iv)	$\frac{1}{\alpha^3\beta^3} + \frac{1}{\beta^3\gamma^3} + \frac{1}{\alpha^3\gamma^3}$ [1 mark] $= \frac{\gamma^3 + \alpha^3 + \beta^3}{\alpha^3\beta^3\gamma^3}$ $= \frac{\sum \alpha^3}{(\alpha\beta\gamma)^3}$ $\sum \alpha^3 = (\sum \alpha)^3 - 3(\sum \alpha)(\sum \alpha\beta) + 3\alpha\beta\gamma$ [1 mark] $= 2^3 - 3(2)\left(\frac{5}{2}\right) + 3\left(\frac{3}{2}\right)$ $= -\frac{5}{2}$ [1 mark] $\frac{1}{\alpha^3\beta^3} + \frac{1}{\beta^3\gamma^3} + \frac{1}{\alpha^3\gamma^3} = -\frac{\frac{5}{2}}{\left(\frac{3}{2}\right)^3} = -\frac{10}{27}$ [1 mark]	4
		TOTAL	25
		Specific Objectives: 2.4, 2.5, 4.1, 4.5, 5.4, 5.5	

PURE MATHEMATICS
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KEY AND MARK SCHEME

	Question	Solutions	Total
3.	(a)	$\frac{\sin 3\theta + \sin 5\theta}{\cos 3\theta + \cos 5\theta}$ $= \frac{2 \sin\left(\frac{3\theta+5\theta}{2}\right) \cos\left(\frac{3\theta-5\theta}{2}\right)}{2 \cos\left(\frac{3\theta+5\theta}{2}\right) \cos\left(\frac{3\theta-5\theta}{2}\right)}$ $= \frac{\sin 4\theta}{\cos 4\theta}$ $= \tan 4\theta$	4
	(b) (i)	$2 \sin \theta - \cos \theta = 0$ $\cos \theta = \frac{\cos \theta}{\sin \theta}$ $2 \sin \theta - \frac{\cos \theta}{\sin \theta} = 0$ $(x \sin \theta)$ $2 \sin^2 \theta - \cos \theta = 0$ $2(1 - \cos^2 \theta) - \cos \theta = 0$ $2 \cos^2 \theta + \cos \theta - 2 = 0$	3
	(ii)	$2 \sin^2 \theta + \cos \theta - 2 = 0$ $(2 \cos \theta - 1)(\cos \theta + 2) = 0$ $\cos \theta = \frac{1}{2} \text{ or } \cos \theta = -2$ $\theta = 60^\circ, \dots$ $\theta \text{ has no solution}$ $\theta = 360n \pm 60^\circ$	4

PURE MATHEMATICS
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	Question	Solutions	Total
	(c) (i)	$\sqrt{3} \sin \theta + \cos \theta = R \sin(\theta + \alpha)$ Equating coefficients: $\cos \theta : R \sin \alpha = 1 \quad [1 \text{ mark}]$ $\sin \theta : R \cos \alpha = \sqrt{3}$ $\tan \alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6} \quad [1 \text{ mark}]$ $R^2 = \sqrt{(\sqrt{3}^2) + 1^2} = 2 \quad [1 \text{ mark}]$ $\sqrt{3} \sin \theta + \cos \theta = 2 \sin\left(\theta + \frac{\pi}{6}\right)$	3
	(ii)	$-1 \leq \sin\left(\theta + \frac{\pi}{6}\right) \leq 1 \quad [1 \text{ mark}]$ $(\times 2) - 2 \leq 2 \sin\left(\theta + \frac{\pi}{6}\right) \leq 2 \quad [1 \text{ mark}]$ $(+3) 1 \leq 2 \sin\left(\theta + \frac{\pi}{6}\right) + 3 \leq 5 \quad [1 \text{ mark}]$ Inverting: $\frac{1}{5} \leq \frac{1}{2 \sin\left(\theta + \frac{\pi}{6}\right) + 3} \leq 1 \quad [1 \text{ mark}]$ Hence the maximum value of $f(\theta) = 1$ [1 mark]	5

PURE MATHEMATICS
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	Question	Solutions	Total
	(d)	$l_1:r=\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}+\lambda\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \quad l_2:r=\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}+\mu\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ Expressing in parametric form: $1-\lambda=3+6\mu \quad (1) \quad [1 \text{ mark}]$ $3\lambda=3+2\mu \quad (2)$ $5-\lambda=4+3\mu \quad (3)$ $(3)-(2)$ $4=1-3\mu \quad [1 \text{ mark}]$ $\mu=-1$ Substitute into (1): $\lambda=4$ [1 mark] Substituting for λ and μ into equation (2) [1 mark] $3(4)=3+2(-1) \quad [1 \text{ mark}]$ $12 \neq 1$ Since there are no unique values of λ and μ which satisfy all three equations, the lines l_1 and l_2 do not intersect. [1 mark]	6
		TOTAL	
		Specific Objectives: 1.1, 1.5, 1.6, 1.7, 1.8, 1.9, 1.11, 3.7, 3.8	

PURE MATHEMATICS
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KEY AND MARK SCHEME

	Question	Solutions	Total
4.	(a)	$x = 3\sin\theta + 5$ $\frac{x-5}{3} = \sin\theta$ [1 mark] $y = 2\cos\theta + 6$ $\frac{y-6}{2} = \cos\theta$ [1 mark] $\left(\frac{x-5}{3}\right)^2 + \left(\frac{y-6}{2}\right)^2 = 1$ $(\cos^2\theta + \sin^2\theta = 1)$ [1 mark]	3
	(b) (i)	Let P have coordinates (x, y). $\frac{ BP = \sqrt{5}}{\sqrt{(x-3)^2 + (y-4)^2}} = \sqrt{5}$ [1 mark] Squaring both sides $(x-3)^2 + (y-4)^2 = 5$ [1 mark] The locus of P is a circle with centre $(3, 4)$ and radius $\sqrt{5}$ units. [1 mark]	3
	(ii)	Gradient of normal = $\frac{5-4}{5-3} = \frac{1}{2}$ [1 mark] Equation of normal: $y-5 = \frac{1}{2}(x-5)$ [1 mark] $y = \frac{1}{2}x + \frac{5}{2}$ Gradient of tangent = -2 [1 mark] Equation of tangent: $y-5 = -2(x-5)$ [1 mark] $y = -2x + 15$	4

PURE MATHEMATICS
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	Question	Solutions	Total
	(iii)	Let $(x-3)^3 + (y-4)^2 = 5 \quad (1)$ $3y = -x + 10 \quad (2)$ From (2) $x = 10 - 3y$ Substitute into (1) $(10 - 3y - 3)^2 + (y - 4)^2 = 5$ $(7 - 3y)^2 + (y - 4)^2 = 5$ $10y^2 + 50y + 60 = 0$ $y^2 + 5y + 6 = 0 \quad [1 \text{ mark}]$ $(y - 3) + (y - 2) = 0$ $y = 2 \text{ or } 3 \quad [1 \text{ mark}]$ When $y = 2, x = 4$ [1 mark] When $y = 3, x = 1$ Coordinate of the points of intersection (1,3) and (4,2) [1 mark]	4

PURE MATHEMATICS
UNIT 1 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(c) (i)	$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \overrightarrow{OB} = \begin{pmatrix} -2 \\ -1 \\ -4 \end{pmatrix} \overrightarrow{OC} = \begin{pmatrix} -4 \\ 2 \\ -5 \end{pmatrix}$ $\overrightarrow{BA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad [1 \text{ mark}]$ $\overrightarrow{CA} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \quad [1 \text{ mark}]$ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix} = 11 + 10 - 21 = 0 \quad [1 \text{ mark}]$ $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix} = 33 - 5 - 28 = 0 \quad [1 \text{ mark}]$ Since both dot products = 0 and both vectors $\overrightarrow{BA}$ and $\overrightarrow{CA}$ lie on the plane, $\begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix}$ is perpendicular to the plane. [1 mark]	5
	(ii)	Using $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ [1 mark] $r \begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix}$ Hence the cartesian equation of the plane is $11x + 5y - 7z = 1$. [1 mark]	3

PURE MATHEMATICS
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	Question	Solutions	Total
	(iii)	Let θ be the angle between the planes π_1 and π_2. $\cos \theta = \frac{n_1 \cdot n_2}{ n_1 n_2 }$ $= \frac{\begin{pmatrix} 11 \\ 5 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}}{\sqrt{195}\sqrt{9}}$ $\theta = 87.3^\circ$	[1 mark] [1 mark] [1 mark] 3
		TOTAL	25
		Specific Objectives: 2.3, 2.4, 2.5, 3.1, 3.9	

PURE MATHEMATICS
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	Question	Solutions	Total
5.	(a) (i)	$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \left(\frac{2x^2 - 4x + 5}{5x} + \frac{5x - 4}{x} \right)$ [1 mark] $= \left(\frac{2(1)^2 - 4(1) + 5}{5(1)} + \frac{5(1) - 4}{1} \right)$ [1 mark] $= \frac{8}{5}$ [1 mark]	3
	(ii)	First principle: [1 mark] $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ [1 mark] $f(x) = x^3 - 6x$ $f(x+h) = (x+h)^3 - 6(x+h)$ [1 mark] $f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^3 - 6(x+h) - (x^3 - 6x)}{h}$ [1 mark] Expand and simplify $f'(x) = \lim_{h \rightarrow 0} \frac{3x^2 + 3xh + h^2 - 6h}{h}$ [1 mark] Divide through by h: $f'(x) = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 6)$ [1 mark] With limit applied $f'(x) = 3x^2 - 6$ [1 mark]	5

PURE MATHEMATICS
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	Question	Solutions	Total
	(b)	Integrate: $g''(x) = 36x^2 - 10$ [1 mark] $g''(x) = 12x^2 - 10x + c$ [1 mark] Using $g'(0), c = 0$ Therefore, $g'(x) = 12x^3 - 10x$ Integrate $g'(x) \Rightarrow g(x) = 3x^4 - 5x^2 + c$ [1 mark] Using $g(1) = 4$ [1 mark] $3(1)^4 - 5(1)^2 + c = 4$ [1 mark] $c = 6$ [1 mark] $g(x) = 3x^4 - 5x^2 + 6$ [1 mark] Hence: $a = 3, b = -5$ [1 mark]	7
	(c) (i)	$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ [1 mark] Since $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$ [1 mark] $\frac{dV}{dt} = d\pi r^2 \times 1.5 = \frac{6\pi r^2 cm^3}{s}$ [1 mark] At $r = 5, \frac{dV}{dt} = 6\pi(5)^2 = 150\pi \frac{cm^2}{s}$ [1 mark]	5

PURE MATHEMATICS
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	Question	Solutions	Total
	(ii)	$\int_0^{2500} dv = \int_0^t 150\pi \, dt$ [1 mark] [1 mark] When $2500 = 150\pi t \Rightarrow t = 5.3s$. [1 mark] $r = \sqrt[3]{\frac{2500 \times 3}{4\pi}} = 8.43 \, cm$ [1 mark] [1 mark]	5
		TOTAL	25
		Specific Objectives: 1.4, 2.2, 2.4, 2.7, 2.10	

PURE MATHEMATICS
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KEY AND MARK SCHEME

	Question	Solutions	Total
6.	(a) (i)	Product rule: $(fg)' = f'g + fg'$ $f = 2x^3 + 3$ $f' = 6x^2$ [1 mark] $g = \cos(x^2)$ Apply Chain rule Let $u = x^2$ and $g = \cos(u)$ [1 mark] $g'(u) = -\sin(u)$ $u' = 2x$ $g' = -2x \sin(x^2)$ [1 mark] Apply Product rule: $(fg)' = 6x^2 \cos(x^2) - 2x(2x^3 + 3)\sin(x^2)$ [1 mark]	4
	(b) (i)	Area of rectangle region between $x = -5$ and $x = 0$ under the line $y = 3$: $A = LW = 5 \times 3 = 15 \text{ squared units}$ [1 mark] Area under curve: $\int_{-5}^0 \sqrt{4-x} dx$ [1 mark] Apply substitution: $U = 4 - x$ Boundaries: $U = 4$ and $U = 9$ [1 mark] Integrating: $-\int_4^9 \sqrt{u} du = \left[\frac{3}{2} u^{3/2} \right]_4^9 = \frac{38}{2} \text{ squared units}$ [2 marks] Area of shaded region $= 15 - \frac{38}{2} = \frac{7}{2}$ squared units [1 mark] [The answer should be given in squared units]	6

PURE MATHEMATICS
UNIT 1 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(ii)	Volume of revolution: $\int_{-5}^0 \pi (\sqrt{4-x-3})^2 dx = \int_{-5}^0 \pi [-x - 6\sqrt{4-x} + 13] dx \quad [1 \text{ mark}]$ $= \pi \left[-\frac{x^2}{2} + 4(4-x)^{3/2} + 13x \right]_{-5}^0 \quad [2 \text{ marks}]$ $= \pi \left[-\frac{25}{2} - 76 + 65 \right] \quad [1 \text{ mark}]$ Volume = $\frac{3}{2}\pi$ cubic units [1 mark] [The answer should be given in cubic units]	5
	(c)	$\frac{d}{dx}(f(x)) = 3x^2 - 7x + 2 \quad [1 \text{ mark}]$ $3x^2 - 7x + 2 = 0 \quad [1 \text{ mark}]$ $x = -\frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(2)}}{2(3)}$ $x = 2, x = \frac{1}{3} \quad [1 \text{ mark}]$ When $x = 2, f(x) = -2 \Rightarrow (2, -2)$ [1 mark] When $x = \frac{1}{3}, f(x) = \frac{17}{54} = 0.315 \Rightarrow \left(\frac{1}{3}, \frac{17}{54}\right)$ [1 mark] $\frac{d}{dx}(3x^2 - 7x + 2) = 6x - 7 \quad [1 \text{ mark}]$ At $x = 2; \frac{d^2}{dx^2}(f(x)) > 0$ $(2, -2)$ is a minimum point [2 marks] At $x = \frac{1}{3}; \frac{d^2}{dx^2}(f(x)) < 0$ $\left(\frac{1}{3}, \frac{17}{54}\right)$ is a maximum point [2 marks]	10
		TOTAL	25
		Specific Objectives: 2.9, 2.12, 2.15, 3.9, 3.10	

SPECIMEN 2022



TEST CODE **02134032**

C A R I B B E A N E X A M I N A T I O N S C O U N C I L

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S P E C I M E N P A P E R

P A P E R 032

2 hours

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE questions.
2. Answer ALL questions.
3. Write your answers in the spaces provided in this booklet.
4. Do NOT write in the margins.
5. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
6. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

***The questions on this paper may be based on Specific Objectives taken from ANY Module in the Unit.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2022**

Mathematical instruments

Silent, non-programmable electronic calculator

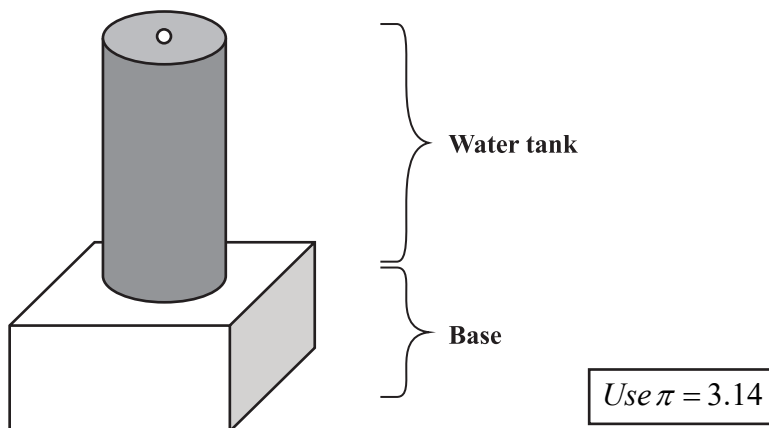
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02134032/SPEC/CAPE 2022

“*”Barcode Area*”
Sequential Bar Code

1. A closed cylindrical water tank is to be installed in a community to provide the residents with a constant supply of water. The diagram below, **not drawn to scale**, shows the water tank mounted on a concrete base.



The population of the community is 400 persons with each person using an average of 0.05 m^3 of water each day

- (a) Calculate, in cubic metres, the minimum volume of water required in the tank daily.

[1 mark]

The manufacturer of the tank can produce a closed cylindrical tank with a volume of 30 m^3 and radius, $x\text{m}$.

- (b) Write an expression, in terms of x , for the height, h , of the tank.

[1 mark]

GO ON TO THE NEXT PAGE

- (c) Write an expression, in terms of x , for the total surface area of the tank.

[2 marks]

- (d) Determine the value of x that would **minimize** the surface area of the tank. Verify that this value of x produces a minimum surface area.

[7 marks]

- (e) The **volume** of water in the tank is being consumed at a rate of $1 \text{ m}^3\text{h}^{-1}$. At what rate is the level of the water in the tank decreasing?

[4 marks]

GO ON TO THE NEXT PAGE

The rate at which the **height** of the water in the tank decreases is given by $\frac{dh}{dt} = -\frac{1}{h^2}$.

- (f) Determine an expression for the height of the water in the tank in terms of time (hours).

[2 marks]

- (g) Initially, the water in the tank is filled to a height of 4 m. Calculate the time it takes for the water level to drop by 2 m.

[3 marks]

Total 20 marks

GO ON TO THE NEXT PAGE

2. Tropical Cruises is a company which operates a fleet of yachts and is interested in offering party cruises between the islands of Barbados, Grenada and St Vincent.

It was determined that the total revenue that could be generated from ticket sales is given by the equation $R(x) = 196x - 3x^2$ where x represents the number of passengers (the level of demand).

- (a) Show that the marginal revenue is given by $R'(x) = 196 - 6x$.

[1 mark]

- (b) Hence, determine the marginal revenue in (hundreds of dollars) when the level of demand, x is 30.

[1 mark]

The total cost function for Tropical Cruises is modelled by $C(x) = 14 + 4x$.

- (c) Show that the profit function for this venture is given by $P(x) = -3x^2 + 192x - 14$.

[3 marks]

GO ON TO THE NEXT PAGE

- (d) Determine an expression for the marginal profit.

[2 marks]

- (e) Compute the marginal profit when the level of demand, x , is 14.

[1 mark]

- (f) Determine whether the profit is increasing or decreasing when the level of demand is 14.

[1 mark]

- (g) Determine the level of demand that maximizes the profit, that is, the value of x for which the marginal profit is equal to zero.

[2 marks]

- (h) Verify that the level of demand stated in (g) is a maximum.

[2 marks]

GO ON TO THE NEXT PAGE

The marginal cost function $C'(x)$ of a rival tourist entity, Wadadli Cruises, is given by $C'(x) = 16x + 10$.

- (i) Write an expression for the total cost function $C(x)$ for Wadadli Cruises.

[3 marks]

- (j) Hence, calculate the value of the constant of integration (overhead cost) for the total cost in (i), given the total cost $C(x) = 32$ when $x = 0$.

[1 mark]

- (k) Which of the two companies, Tropical Cruises or Wadadli, would have the lower overhead cost? **Justify your response.**

[3 marks]

Total 20 marks

GO ON TO THE NEXT PAGE

3. The table below shows the ports visited by Tropical Cruises in the respective islands.

Island	Port	Location	Height Above Sea Level (m)
Barbados	Bridgetown (B)	13.1° North, 59.6° West	1
Grenada	Prickly Bay (P)	12° North, 61.8° West	21
St Vincent and the Grenadines	Kingstown (K)	13.2° North, 61.2° West	14

*To the nearest metre

- (a) If $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ represents a port located at x° north and y° west and z metres above sea level, express the location of the ports B , P and K as position vectors.

[3 marks]

- (b) In the space provided below, sketch the relative positions of the three ports B , P and K as would be seen on a map.

[3 marks]

GO ON TO THE NEXT PAGE

(c) Determine the vector equation of the line joining EACH of the following ports:

(i) B and P

[2 marks]

(ii) B and K

[2 marks]

(d) Calculate the acute angle between the lines BP and BK .

[4 marks]

(e) Show that the vector $11\mathbf{i} + 15.4\mathbf{j} - 1.98\mathbf{k}$ is perpendicular to both vectors, $\overrightarrow{BP}$ and $\overrightarrow{BK}$.

[3 marks]

GO ON TO THE NEXT PAGE

- (f) Hence, determine the Cartesian equation of the plane containing the points Prickly Bay, Bridgetown and Kingstown.

[3 marks]

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

02134032/SPEC/CAPE 2022

“*”Barcode Area”*”
Sequential Bar Code

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[illegible]

Question No.

“*”Barcode Area”*”

Sequential Bar Code

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Barcode Area
Sequential Bar Code

“*”Barcode Area”*”
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CANDIDATE'S RECEIPT

INSTRUCTIONS TO CANDIDATE:

1. Fill in all the information requested clearly in capital letters.

TEST CODE:

0	2	1	3	4	0	3	2
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SUBJECT: PURE MATHEMATICS – UNIT 1 – Paper 032

PROFICIENCY: ADVANCED

REGISTRATION NUMBER:

--	--	--	--	--	--	--	--	--	--

FULL NAME: _____
(BLOCK LETTERS)

Signature: _____

Date: _____

2. Ensure that this slip is detached by the Supervisor or Invigilator and given to you when you hand in this booklet.
3. Keep it in a safe place until you have received your results.

INSTRUCTION TO SUPERVISOR/INVIGILATOR:

Sign the declaration below, detach this slip and hand it to the candidate as his/her receipt for this booklet collected by you.

I hereby acknowledge receipt of the candidate's booklet for the examination stated above.

Signature: _____
Supervisor/Invigilator

Date: _____

Barcode Area
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C A R I B B E A N E X A M I N A T I O N S C O U N C I L

CARIBBEAN ADVANCED PROFICIENCY EXAMINATIONS®

PURE MATHEMATICS

UNIT 1 – Paper 032

KEY AND MARK SCHEME

MAY/JUNE 2022

SPECIMEN PAPER

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Questions	Solutions	Total
1.	(a)	Minimum volume of water $= 400 \times 0.05$ $= 20m^3$ [1 mark]	1
	(b)	$V = \pi r^2 h$ $h = \frac{V}{\pi r^2}$ Substituting $V = 30$ and $r = x$ $h = \frac{30}{\pi x^2}$ making h the subject of the formula. [1 mark]	1
	(c)	The surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$ Substituting $h = \frac{30}{\pi x^2}$ and $r = x$ $A = 2\pi r^2 + 2\pi x \left(\frac{30}{\pi x^2} \right)$ Substituting into formula $= 2\pi r^2 + \frac{60}{x}$ simplifying expression. [2 marks]	2

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Questions	Solutions	Total
	(d)	$A = 2\pi x^2 + \frac{60}{x}$ $\frac{dA}{dx} = 4\pi x - \frac{60}{x^2}$ Differentiating [1 mark] Stationary points occur when $\frac{dA}{dx} = 0$ i.e. $0 = 4\pi x - \frac{60}{x^2}$ placing $\frac{dA}{dx} = 0$. $\frac{60}{x^2} = 4\pi x$ [1 mark] $\frac{15}{\pi} = x^3$ [1 mark] $\sqrt[3]{\frac{15}{\pi}} = x$ [1 mark] find x $x = 1.683$ [1 mark] $\approx 1.68m^3 (3sf)$ units $\frac{d^2A}{dx^2} = 4\pi + \frac{120}{x^3}$ finding the 2nd derivative [1 mark] When $x = 1.68$ $\frac{d^2A}{dx^2} > 0$ 2nd derivative sign test. Hence the area is a minimum. [1 mark]	7

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Questions	Solutions	Total
	(e)	$V = \pi r^2 h$ $\frac{dV}{dh} = \pi r^2$ [1 mark] Differentiation $= \pi (1.68)^2$ $= 8.86 \text{ (3 s.f.)}$ [1 mark] Substituting and evaluating $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ Correct use of chain rule [1 mark] $= \frac{1}{8.86} \quad (1)$ $= 0.1129$ $\approx 0.13 \text{ mh}^{-1}$ Obtaining correct value of $\frac{dh}{dt}$. [1 mark]	4
	(f)	The rate at which the height of the water in the tank decreases is given by $\frac{dh}{dt} = -\frac{1}{h^2}$ $h^2 dh = -dt$ Separating variables [1 mark] The variables are separable and the solution can be found by integration. $-\int h^2 dh = \int dt$ $-\frac{h^3}{3} = t + c$ Integrating correctly [1 mark] $h = -\sqrt[3]{3(t+c)}$	2

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Questions	Solutions	Total
	(g)	When, $t = 0$, $h = 4$ substituting the initial conditions. $-\frac{4^3}{3} = 0 + c$ $-\frac{64}{3} = c$ Obtaining the value of the constant [1 mark] When $h = 2$ $-\frac{2^3}{3} = t - \frac{64}{3}$ $\frac{64}{3} - \frac{8}{3} = t$ $t = \frac{56}{3} \text{ hours}$ Obtaining the time t. [1 mark]	3
		Specific Objective Module 2.2, 2.3, 2.4, 2.7, 2.9; 2.10; 2.11; 2.12, 3.2; 3.4; 3.12; 3.13	

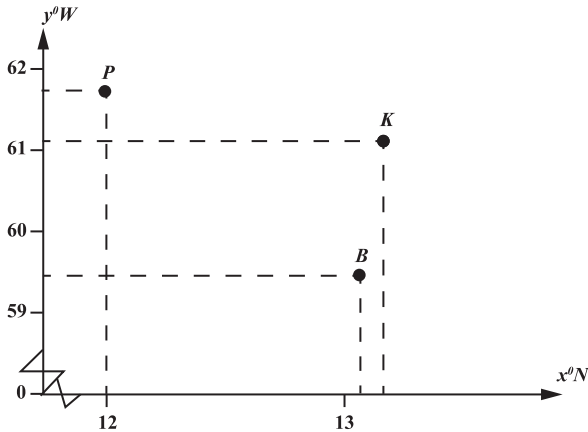
PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Questions	Solutions	Total
2.	(a)	$R(x) = 196x - 3x^2$ $R'(x) = 196 - 2(3x)$ $= 196 - 6x$ [1 mark]	1
	(b)	When demand level $x = 30$ Marginal revenue = $196 - 6(30)$ $= 196 - 180 = 16$ (hundred dollars) [1 mark]	1
	(c)	$C(x) = 14 + 4x$ $P(x) = R(x) - C(x)$ [1 mark] $= 196x - 3x^2 - (14 + 4x)$ [1 mark] $= 196x - 3x^2 - 14 - 4x$ $P(x) = -3x^2 + 192x - 14$ [1 mark]	3
	(d)	Marginal profit function $P'(x) = 2(-3x) + 192$ [1 mark] $P'(x) = -6x + 192$ [1 mark]	2
	(e)	When demand ($x = 14$) Marginal Profit = $-6(14) + 192$ $= -84 + 192$ $= 108$ (hundred dollars) [1 mark]	1
	(f)	Since Marginal profit > 0 then profit is increasing [1 mark]	1
	(g)	$= -6x + 192 = 0$ [1 mark] $= -6x = -192$ $x = 32$ [1 mark] The level of demand is 32 passengers	2

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Questions	Solutions	Total
	(h)	$P''(x) = -6 < 0$ [1 mark] Hence at $x = 32$, $P(x)$ is maximum. [1 mark]	2
	(i)	$C(x) = \int (16x + 10) dx$ [1 mark] $= \frac{1}{1+1} (16x^{1+1}) + 10x + c$ [1 mark] $= \frac{1}{2} (16x^2) + 10x + c$ $= 8x^2 + 10x + c$ [1 mark]	3
	(j)	$C(x) = 32$ when $x = 0$. $8x^2 + 10x + c = 32$ When $x = 0$. $8(0)^2 + 10(0) + c = 32$ $0 + 0 + c = 32$ $c = 32$ [1 mark]	1
	(k)	By comparison, Total Cost function for Tropical Cruises is $C(x) = 14 + 4x$. While the total cost function for Wadadli Cruises is $C(x) = 8x^2 + 10x + c$. The overhead cost (when $x = 0$) for Tropical Cruises is $C(x) = 14 + 4(0) = 14$ and the overhead cost (when $x = 0$) for Wadadli Cruises is $C = 32$ from part (x). Hence, by comparison, Tropical Cruises would have the lower overhead cost. [1 mark]	3
		Specific Objective Module: 2.2, 2.3, 2.4, 2.8; 2.9; 2.10; 3,1; 3.2; 3.3; 3.4	

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Questions	Solutions	Total
3.	(a)	$\overrightarrow{OB} = 13.1i + 59.6j + k$ [1 mark] $\overrightarrow{OP} = 12i + 61.8j + 21k$ [1 mark] $\overrightarrow{OK} = 13.2i + 61.2j + 14k$ [1 mark]	3
	(b)	 1 mark each for plotting the points B, K and P	3
	(c) (i)	$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$ [1 mark] $= 1.1i + 2.2j + 20k$ The equation of the line, $BP, r = \begin{pmatrix} 13.1 \\ 59.6 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1.1 \\ 2.2 \\ 11 \end{pmatrix}$ [1 mark]	2
	(ii)	$\overrightarrow{BK} = \overrightarrow{OK} - \overrightarrow{OB}$ [1 mark] $= 0.1i + 1.6j + 13k$ [1 mark] $\begin{pmatrix} 13.1 \\ 59.6 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0.1 \\ 1.6 \\ 13 \end{pmatrix}$ [1 mark]	2

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total
	(d)	Let θ represent the angle between the lines BP and BK. $\cos \theta = \frac{\begin{pmatrix} -1.1 \\ 2.2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 0.1 \\ 1.6 \\ 13 \end{pmatrix}}{\sqrt{(-1.1)^2 + (2.2)^2 + (11)^2} \sqrt{(0.1)^2 + (1.6)^2 + (13)^2}}$ $= \frac{146.63}{147.64}$ $\theta = \cos^{-1}\left(\frac{146.41}{147.64}\right)$ $= 7.40^\circ$ [1 mark] Correct formula [1 mark] Correct numerator [1 mark] Correct denominator [1 mark]	4
	(e)	$\begin{pmatrix} -1.1 \\ 2.2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 15.4 \\ -1.98 \end{pmatrix} = 0$ Showing dot product = 0 [1 mark] $\begin{pmatrix} 0.1 \\ 1.6 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 15.4 \\ -1.98 \end{pmatrix} = 0$ Showing dot product = 0 [1 mark] Since dot product = 0 in both cases, the vector $11i + 15.4j - 1.98k$ is perpendicular to both vectors $\overrightarrow{BP}$ and $\overrightarrow{BK}$. Stating reason for being perpendicular. [1 mark]	3

PURE MATHEMATICS
UNIT 1 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total
	(f)	Since $\begin{pmatrix} 11 \\ 15.4 \\ -1.98 \end{pmatrix}$ is perpendicular to both $\overrightarrow{BP}$ and $\overrightarrow{BK}$, then this vector is perpendicular to the plane containing all 3 ports. The equation of the plane containing the ports B, P and K is given by $r.n=a.n$ Correct formula $r \begin{pmatrix} 11 \\ 15.4 \\ -1.98 \end{pmatrix} = \begin{pmatrix} 13.1 \\ 59.6 \\ 1 \end{pmatrix} \begin{pmatrix} 11 \\ 15.4 \\ -1.98 \end{pmatrix} \quad [1 \text{ mark}]$ Correct substitution [1 mark] $11x + 15.4y - 1.98z = (13.1)(11) + (59.6)(15.4) + (1)(-1.98)$ $11x + 15.4y - 1.98z = 1059.96 \quad [1 \text{ mark}]$ Stating the Cartesian equation	3
		Specific Objective Module: 3.1; 3.2; 3.3; 3.6; 3.7; 3.9	



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PURE MATHEMATICS

COMPLEX NUMBERS, ANALYSIS AND MATRICES

UNIT 2 – Paper 01

90 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This test consists of 45 items. You will have 90 minutes to answer them.
2. Each item in this test has four suggested answers lettered (A), (B), (C), (D). Read each item you are about to answer and decide which choice is best.
3. Look at the sample item below.

Sample ItemIf the function $f(x)$ is defined by $f(x) \cos x$ then $f(x)$ isSample Answer

(A) $-\frac{1}{2\sqrt{x}} \sin \sqrt{x}$

(B) $-\frac{1}{2} \sin \sqrt{x}$

(C) $\frac{1}{\sqrt{x}} \sin \sqrt{x}$

(D) $\frac{1}{2\sqrt{x}} \sin \sqrt{x}$



The best answer to this item is “ $\frac{1}{2\sqrt{x}} \sin \sqrt{x}$ ”, so answer space (D) has been shaded.

4. You may do any rough work in this booklet.
5. The use of silent, non-programmable scientific calculators is allowed.

Examination Materials Permitted

A list of mathematical formulae and tables (provided) — Revised 2012

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

1. If z and z^* are two conjugate complex numbers, where $z = x + iy$, $x, y \in \mathbb{R}$, then $z z^* =$
- (A) $x^2 + y^2$
(B) $x^2 - y^2 - 2xyi$
(C) $x^2 - y^2$
(D) $x^2 + y^2 - 2xyi$
2. If $|z + i| = |z + 1|$, where z is a complex number, then the locus of z is
- (A) $y = 0$
(B) $y = 1$
(C) $y = x$
(D) $y = \frac{x}{2}$
3. Given that $z + 3z^* = 12 + 8i$, then $z =$
- (A) $-3 - 4i$
(B) $3 - 4i$
(C) $3 + 4i$
(D) $-3 + 4i$
4. The gradient of the normal to the curve with the equation $xy^3 + y^2 + 1 = 0$ at the point $(2, -1)$ is
- (A) -4
(B) $-\frac{1}{4}$
(C) $\frac{1}{4}$
(D) 4
5. If $\frac{dy}{dx} = \frac{5x}{y}$, then $\frac{d^2y}{dx^2} =$
- (A) $\frac{5}{y^3} - \frac{25x^2}{y^2}$
(B) $\frac{5}{y} - \frac{25x^2}{y^3}$
(C) $\frac{1}{y} - \frac{25}{xy}$
(D) $\frac{5}{xy} - \frac{25}{x^3y^3}$
6. Given $y = a \arccos(ax)$, where a is a constant, $\frac{dy}{dx} =$
- (A) $\frac{a^2}{\sqrt{(1-a^2x^2)}}$
(B) $-\frac{1}{\sqrt{(1-a^2x^2)}}$
(C) $-\frac{a^2}{\sqrt{(1-a^2x^2)}}$
(D) $\frac{1}{\sqrt{(1-a^2x^2)}}$
7. Given that $f(x, y, z) = x^2y + y^2z - z^2x$ then $\frac{\partial f}{\partial y} =$
- (A) $x^2y + 2yz$
(B) $x^2 + 2yz$
(C) $x^2 + y^2$
(D) $x^2 + y^2 + z^2$

GO ON TO THE NEXT PAGE

8. $\int_{\text{as}} \left[x + 3 + \frac{M}{x^2 - 3x + 2} \right] dx$ may be expressed as
- (A) $\int \left[\frac{Px + Q}{x^2 - 3x + 2} \right] dx$
- (B) $\int \left[\frac{P}{x - 1} + \frac{Q}{x - 2} \right] dx$
- (C) $\int \left[x + 3 + \frac{Px + Q}{x^2 - 3x + 2} \right] dx$
- (D) $\int \left[x + 3 + \frac{P}{x - 1} + \frac{Q}{x - 2} \right] dx$
9. Given $I_n = \int \tan^n x \, dx$, for $n > 2$, $I_n =$
- (A) $\frac{1}{n - 1} \tan^{n-1} x + I_{n-2}$
- (B) $\frac{1}{n - 1} \tan^{n-1} x \sec^2 x - I_{n-2}$
- (C) $\tan^{n-1} x - I_{n-2}$
- (D) $\frac{1}{n - 1} \tan^{n-1} x - I_{n-2}$
10. $\frac{d}{dx} (e^{3x^2 + 2x + 1})$ is
- (A) $(6x + 2) e^{6x + 2}$
- (B) $(6x + 2) e^{3x^2 + 2x + 1}$
- (C) $(3x^2 + 2x + 1) e^{6x + 2}$
- (D) $(3x^2 + 2x + 1) e^{3x^2 + 2x + 1}$
11. A curve is defined parametrically by the equations $x = t^2$, $y = t(1 - t^2)$. The gradient of the curve, in terms of t , is
- (A) $\frac{1 - 3t^2}{2t}$
- (B) $\frac{2t}{1 - 3t^2}$
- (C) $2t(1 - 2t)$
- (D) $2t(1 + 2t)$
12. Given $y = \ln(2x + 3)^3$, then $\frac{dy}{dx}$ is
- (A) $\frac{2x}{2x + 3}$
- (B) $\frac{2}{2x + 3}$
- (C) $\frac{6x}{2x + 3}$
- (D) $\frac{6}{2x + 3}$
13. $\int x e^{2x} \, dx$ may be expressed as
- (A) $2x e^{2x} + e^{2x} + c$
- (B) $2x e^{2x} - 4e^{2x} + c$
- (C) $\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c$
- (D) $\frac{1}{2} x^2 e^{2x} + \frac{1}{2} x e^{2x} + c$
14. Which of the following functions, when integrated with respect to x , gives the result $x - \ln x^2 + K$?
- (A) $\frac{x - 2}{x}$
- (B) $\frac{1 - 2x}{x^2}$
- (C) $\frac{1}{1 - x^2}$
- (D) $1 - \frac{2}{x^2}$

GO ON TO THE NEXT PAGE

15. Written as partial fractions $\frac{5}{(x+2)(x-3)}$ is

- (A) $\frac{1}{(x+2)} + \frac{1}{x-3}$
 (B) $\frac{-1}{(x+2)} + \frac{1}{x-3}$
 (C) $\frac{1}{(x+2)} + \frac{-1}{x-3}$
 (D) $\frac{-1}{(x+2)} + \frac{-1}{x-3}$

16. Given that a sequence of positive integers $\{U_n\}$ is defined by $U_1 = 2$ and $U_{n+1} = 3U_n + 2$, then $U_n =$

- (A) $3n - 1$
 (B) $3^n + 1$
 (C) $3^n - 1$
 (D) $3n + 2$

17. The sequence $a_n = \frac{3n^2 - n + 4}{2n^2 + 1}$

- (A) converges
 (B) diverges
 (C) is periodic
 (D) is alternating

18. The n th term of a sequence is given by $u_n = 9 - 4 \left(\frac{1}{2} \right)^{n-1}$. The 5th term of the sequence is

- (A) $\frac{9}{4}$
 (B) $\frac{35}{4}$
 (C) $\frac{37}{4}$
 (D) $\frac{71}{8}$

19. Given that $\sum_{r=1}^n u_r = 5n + 2n^2$, then $u_n =$

- (A) $4n + 3$
 (B) $5n + 2$
 (C) $2n^2 + n - 3$
 (D) $4n^2 + 4n + 7$

20. $\sum_{r=1}^{m-1} 3 \left(\frac{1}{2} \right)^r =$

- (A) $3 - 3 \times 2^{-m}$
 (B) $3 - 3 \times 2^{(1-m)}$
 (C) $6 - 3 \times 2^{(m-1)}$
 (D) $6 - 3 \times 2^{(1-m)}$

21. The Maclaurin's series expansion for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$ has a general term BEST defined as

- (A) $(-1)^n \frac{x^{2n+1}}{(2n+1)!}$
 (B) $(-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$
 (C) $\frac{37}{4}$
 (D) $\frac{71}{8}$

22. The first 2 non-zero terms of the expansion of $\sin \left(x + \frac{\pi}{6} \right)$ are

- (A) $\frac{1}{2} + \frac{\sqrt{3}}{2} x$
 (B) $\frac{1}{2} - \frac{\sqrt{3}}{2} x$
 (C) $\frac{1}{2} + \frac{1}{2} x$
 (D) $\frac{\sqrt{3}}{2} + \frac{1}{2} x$

23. If $\left(2x^2 - \frac{2}{x}\right)^6 = \dots + k + \dots$, where k is independent of x , then $k =$
- (A) -960
(B) -480
(C) 480
(D) 960
24. $f(x) = x^3 - x^2 - 6$. Given that $f(x) = 0$ has a real root α in the interval $[2.2, 2.3]$, applying linear interpolation once on this interval an approximation to α , correct to 3 decimal places, is
- (A) 2.216
(B) 2.217
(C) 2.218
(D) 2.219
25. Taking 1.6 as a first approximation to α , where the equation $4 \cos x + e^{-x} = 0$ has a real root α in the interval $(1.6, 1.7)$, using the Newton-Raphson method a second approximation to α (correct to 3 decimal places) $\approx$
- (A) 1.602
(B) 1.620
(C) 1.622
(D) 1.635
26. $f(x) = 3x^3 - 2x - 6$. Given that $f(x) = 0$ has a real root, α , between $x = 1.4$ and $x = 1.45$, starting with $x_0 = 1.43$ and using the iteration $x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{2}{3}\right)}$, the value of x_1 correct to 4 decimal places is
- (A) 1.4369
(B) 1.4370
(C) 1.4371
(D) 1.4372
27. Given $E \equiv \sum_{n=19}^{30} \frac{3^n}{n}$. The number of terms in the expansion of E is
- (A) 10
(B) 11
(C) 12
(D) 13
28. The second and fifth terms of a convergent geometric series with first term $\frac{81}{2}$ are 27 and 8, respectively. The sum to infinity of this series is
- (A) $\frac{2}{3}$
(B) $\frac{3}{2}$
(C) $\frac{81}{2}$
(D) $\frac{243}{2}$

GO ON TO THE NEXT PAGE

29. The first term of an AP is 'a' and its common difference is -1 . The sum of the first 10 terms is equal to
- (A) $5(2a - 9)$
(B) $5(2a + 9)$
(C) $10(2a + 11)$
(D) $10(2a - 11)$
30. The coefficient of x^2 in the expansion of $(2 - 3x)^5$ is
- (A) -720
(B) -240
(C) 240
(D) 720
31. On a randomly chosen day the probability that Bill travels to school by car, by bicycle or on foot is $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively. The probability of being late when using these methods of travel is $\frac{1}{5}$, $\frac{2}{5}$ and $\frac{1}{10}$ respectively. The probability that on a randomly chosen day Bill travels by foot and is late is
- (A) $\frac{1}{30}$
(B) $\frac{1}{10}$
(C) $\frac{3}{10}$
(D) $\frac{13}{30}$
32. Ten cards, each of a different colour, and consisting of a red card and a blue card, are to be arranged in a line. The number of different arrangements in which the red card is not next to the blue card is
- (A) $9! - 2 \times 2!$
(B) $10! - 9! \times 2!$
(C) $10! - 2! \times 2!$
(D) $8! - 2! \times 2!$
33. The number of ways in which all 10 letters of the word **STANISLAUS** can be arranged if the Ss must all be together is
- (A) $\frac{8! \times 3!}{2!}$
(B) $8! \times 3!$
(C) $\frac{8!}{2!}$
(D) $\frac{8!}{3!}$
34. A committee of 4 is to be chosen from 4 teachers and 4 students. The number of different committees that can be chosen if there must be at least 2 teachers is
- (A) 36
(B) 45
(C) 53
(D) 192

GO ON TO THE NEXT PAGE

35. A and B are two events such that

$$P(A) = p \text{ and } P(B) = \frac{1}{3}.$$

The probability that neither occurs is $\frac{1}{2}$.

If A and B are mutually exclusive events then $p =$

- (A) $\frac{5}{6}$
- (B) $\frac{2}{3}$
- (C) $\frac{1}{5}$
- (D) $\frac{1}{6}$

Items **36–37** refer to the matrix.

$$\begin{bmatrix} \textcircled{-2} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

36. The cofactor of the circled element, -2 , is

- (A) -2
- (B) -1
- (C) 0
- (D) 2

37. The determinant of the given matrix is

- (A) -5
- (B) -3
- (C) 3
- (D) 5

38. Given $\begin{vmatrix} 6 & 0 & 1 \\ 7 & 7 & 0 \\ 0 & -12 & x \end{vmatrix} = 0$, the value of x is

- (A) -2
- (B) 2
- (C) 7
- (D) 12

39. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ and

$$B = \begin{bmatrix} -4 & 0 & 2 \\ 0 & 6 & -4 \\ 2 & -4 & 2 \end{bmatrix}, \text{ by considering}$$

AB , then $A^{-1} =$

- (A) $2B$
- (B) B
- (C) $\frac{1}{2} B$
- (D) $\frac{1}{2} AB$

GO ON TO THE NEXT PAGE

40. If $A = \begin{bmatrix} 2 & -7 & 8 \\ 3 & -6 & -5 \\ 4 & 0 & -1 \end{bmatrix}$, the transpose of matrix, A , results in $|A|$ being

(A) zero
(B) squared
(C) negative
(D) unchanged

41. The general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + x - 1$ is

(A) $y = e^{2x} (A + Bx) + 2x^2 + x - 1$
(B) $y = e^{-2x} (A + B) + 2x^2 - x + 1$
(C) $y = e^{2x} (A + Bx) + \frac{x^2}{2} + \frac{5}{4}x + \frac{3}{4}$
(D) $y = e^{2x} (A - Bx) + \frac{x^2}{2} + \frac{5}{4}x + \frac{3}{4}$

42. The general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0 \text{ is}$$

(A) $y = Ae^x + Be^{-2x}$
(B) $y = Ae^x + Be^{2x}$
(C) $y = e^x (C \cos 2x + D \sin 2x)$
(D) $y = e^{2x} (C \cos x + D \sin x)$

43. The general solution of the differential equation $\sin x \frac{dy}{dx} - y \cos x = \sin 2x \sin x$ is found by evaluating

(A) $\int \frac{d}{dx} y \sin x \, dx = \int 2 \cos x \, dx$
(B) $\int \frac{d}{dx} \frac{y}{\sin x} \, dx = \int 2 \cos x \, dx$
(C) $\int \frac{d}{dx} \frac{y}{\sin x} \, dx = \int \sin 2x \, dx$
(D) $\int \frac{d}{dx} \frac{y}{\sin x} \, dx = \int \cos x \, dx$

44. A particular integral of the differential equation $\frac{d^2y}{dx^2} + 25y = 3 \cos 5x$ is of the form $y = \lambda x \sin 5x$. The general solution of the differential equation is

(A) $y = A \cos 5x - B \sin 5x - \lambda x \sin 5x$
(B) $y = A \cos 5x + B \sin 5x + \lambda x \sin 5x$
(C) $y = A \cos 5x + B \sin 5x - \lambda x \sin 5x$
(D) $y = A \cos 5x - B \sin 5x + \lambda x \sin 5x$

45. The general solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3e^x$ is of the form

(A) $y = Ae^x + Be^{2x} + ke^x$
(B) $y = Ae^x + Be^{2x} - 3e^x$
(C) $y = Ae^{-x} + Be^{-2x} + kxe^x$
(D) $y = Ae^x + Be^{2x} + kxe^x$

END OF TEST

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SUBJECT: PURE MATHEMATICS – Unit 1 – Paper 01

PROFICIENCY: ADVANCED

REGISTRATION NUMBER:

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UNIT 2

PAPER 01

Item	Key	Specific Obj.
1	A	1.4
2	C	1.10
3	B	1.4
4	A	2.4
5	B	2.4
6	C	2.5
7	B	2.8
8	D	3.1
9	D	3.9
10	B	2.1
11	A	2.3
12	D	2.2
13	C	3.7
14	A	3.5
15	B	3.1

Item	Key	Specific Obj.
16	C	1.2
17	A	1.3
18	B	1.2
19	A	2.3
20	B	2.3
21	A	2.7
22	A	2.8
23	D	3.2
24	C	4.3
25	B	4.5
26	C	4.6
27	C	2.2
28	D	2.5
29	A	2.2
30	D	3.2

Item	Key	Specific Obj.
31	A	1.12
32	B	1.2
33	D	1.2
34	C	1.3
35	D	1.11
36	A	2.4
37	C	2.3
38	B	2.3
39	C	2.4
40	D	2.2
41	C	3.2
42	C	3.2
43	B	3.1
44	B	3.2
45	D	3.2

SPECIMEN 2022



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PURE MATHEMATICS

UNIT 2

COMPLEX NUMBERS, ANALYSIS AND MATRICES

SPECIMEN PAPER

PAPER 02

2 hours 30 minutes

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE sections.
2. Each section consists of TWO questions.
3. Answer ALL questions from the THREE sections.
4. Write your answers in the spaces provided in this booklet.
5. Do NOT write in the margins.
6. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
7. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
8. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

Examination Materials

Mathematical formulae and tables (provided) – **Revised 2022**

Electronic calculator

Ruler and graph paper

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SECTION A

Module 1

Answer BOTH questions.

1. (a) (i) Express the complex number $\frac{3-4i}{2+i}$ in the form $a + bi$ where a and b are real numbers.

[4 marks]

- (ii) Hence, calculate the principal argument of $\frac{3-4i}{2+i}$.

[3 marks]

- (b) (i) Given that $u^2 = -3 + 4i$, derive the complex numbers of the form $u = x + iy$, where $x, y \in \mathbb{R}$.

[6 marks]

GO ON TO THE NEXT PAGE

- (ii) Hence solve the equation $z^2 - z + (1 - i) = 0$.

[4 marks]

- (c) The complex number $z = x + iy$ satisfies the equation $|z - 3 + 2i| = 2|z - 1 - 4i|$.

The complex number z is represented by the point **P** on an Argand diagram.

- (i) Show that the locus of **P** is a circle.

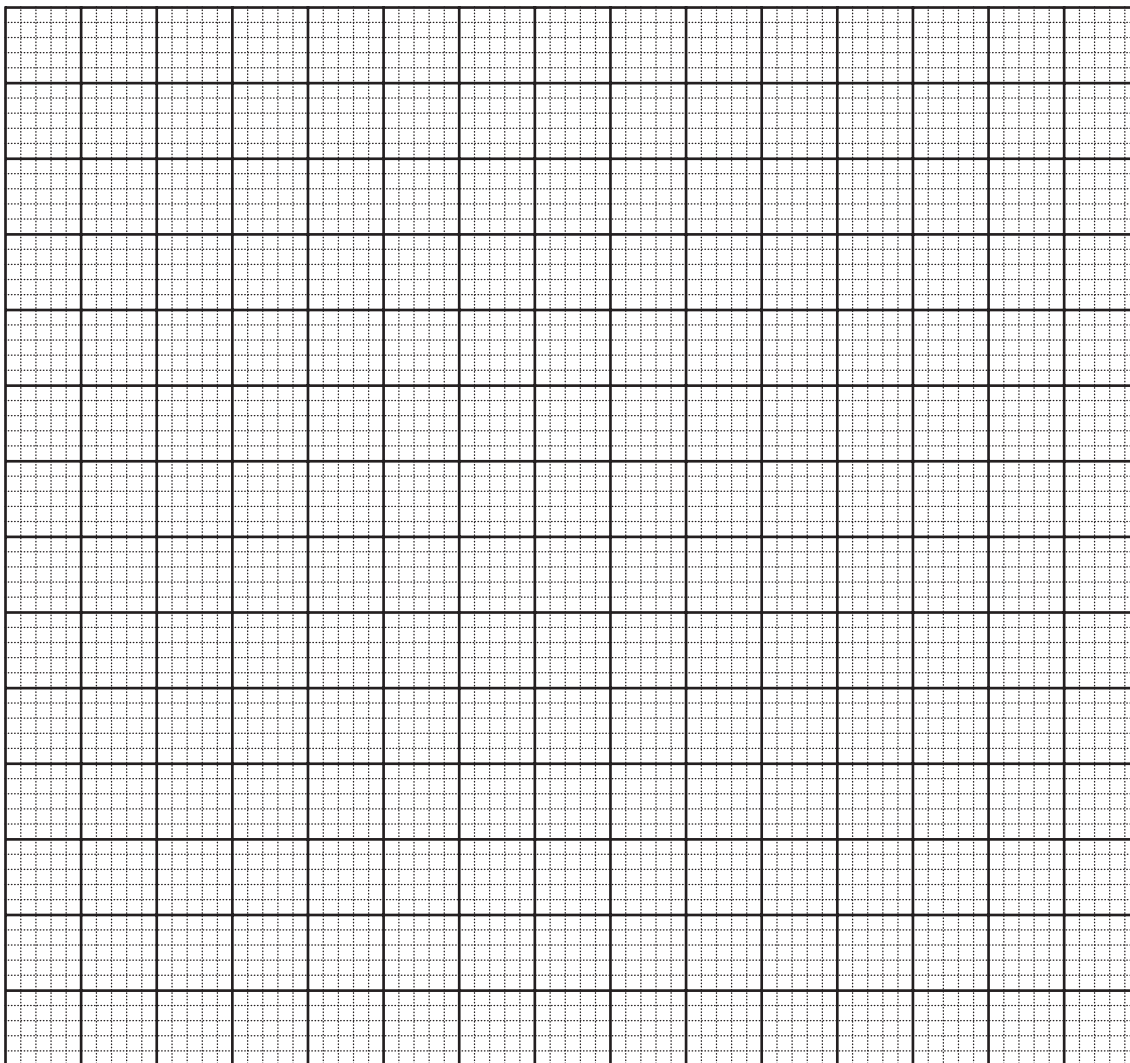
[4 marks]

- (ii) State the centre and radius of the circle.

[2 marks]

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- (iii) Use the grid provided below to sketch the circle on an Argand diagram.



[2 marks]

Total 25 marks

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2. (a) Evaluate $\int e^{2x} \cos 3x \, dx$.

[7 marks]

- (b) (i) Given that $x^2 + 4y^2 = 1$, write an expression for $\frac{dy}{dx}$.

[3 marks]

- (ii) Hence, show that $\frac{d^2y}{dx^2} = \frac{-1}{16y^3}$.

[4 marks]

GO ON TO THE NEXT PAGE

- (c) Use the substitution $u = 1 + x^2$ to evaluate $\int_0^1 \frac{x^3}{(1 + x^2)^4} dx$.

[7 marks]

- (d) Let $f(x, y) = 3x^2 - xy - 3y^2$

- (i) Write expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

[2 marks]

- (ii) Hence, determine the value of $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

[2 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION B

Module 2

Answer BOTH questions.

3. (a) (i) Determine the value of the constants A and B such that

$$\frac{1}{(1-2r)(1+2r)} = \frac{A}{1-2r} + \frac{B}{1+2r}.$$

[5 marks]

- (ii) Hence, determine the value of S where $S = \sum_{r=1}^n \frac{1}{(1-2r)(1+2r)}.$

[5 marks]

- (iii) Deduce the sum to infinity of S .

[3 marks]

GO ON TO THE NEXT PAGE

- (b) (i) Write an expression for the r^{th} term of the series $1(8) + 2(20) + 3(32) + \dots$

[2 marks]

- (ii) Prove, by mathematical induction, that the sum to n terms of the series in (b) (i) is $2n^2(2n + 2)$.

[10 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

4. (a) Given the series $\frac{1}{3} + \frac{1}{3^4} + \frac{1}{3^7} + \frac{1}{3^{10}} + \dots$

(i) show that the series is a geometric series.

[3 marks]

(ii) determine the sum of the first n terms of the series.

[4 marks]

(b) Use Maclaurin's Theorem to find the first THREE **non-zero** terms in the power series expansion of $\sin 2x$.

[7 marks]

GO ON TO THE NEXT PAGE

- (c) (i) Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to and including the term in x^3 stating the values of x for which the expression is valid.

[7 marks]

- (ii) By taking $x = 0.04$, determine an approximation for $\sqrt{\frac{52}{3}}$, correct to 3 decimal places.

[4 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION C

Module 3

Answer BOTH questions.

5. (a) Two fair coins and one fair die are tossed at the same time.

- (i) Calculate the total number of outcomes in the sample space.

[3 marks]

- (ii) Determine the probability of obtaining exactly one head and a 6.

[2 marks]

- (iii) Calculate the probability of obtaining at least one head and an even number on the die, on a particular attempt.

[4 marks]

GO ON TO THE NEXT PAGE

- (b) A record is kept of the books borrowed from a library. The table below shows the number of overdue books outstanding at the end of the week.

	Adult		Child	
	Male	Female	Male	Female
Fiction	17	10	14	20
Non-fiction	9	14	5	11

One of these books has been requested by another reader. Determine the probability that

- (i) it is non-fiction, borrowed by a male child

[2 marks]

- (ii) it is fiction, borrowed by a female

[3 marks]

- (iii) it was borrowed by an adult male, given that it is fiction.

[3 marks]

- (c) On a particular weekend, 100 customers purchased tools (T), fertilizer (F) or Seeds (S) at Green Thumb Garden Supply Store. Of these,

38 purchased tools
56 purchased fertilizer
40 purchased seeds
30 purchased seeds and fertilizer
17 purchased seeds and tools
20 purchased tools and fertilizer
14 purchased tools, seeds and fertilizer

- (i) Represent the information on a Venn diagram.

[3 marks]

- (ii) Using your diagram determine,

- a) the probability that a customer does not purchase seeds or tools

[2 marks]

- b) whether the purchase of seeds and fertilizer are independent.

[3 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

6. (a) A matrix $\mathbf{A}$ is given as
$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -1 & 6 & 0 \end{pmatrix}$$

(i) Calculate the determinant of $\mathbf{A}$.

[5 marks]

(ii) Hence, determine $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$.

[5 marks]

GO ON TO THE NEXT PAGE

- (b) The amount of salt, y kg, that dissolves in a tank of water at time t minutes satisfies the first order differential equation $\frac{dy}{dt} + \frac{2y}{t+10} = 3$. Using a suitable integrating factor, show that the general solution of this differential equation is $y = t + 10 + \frac{c}{(t+10)^2}$, where c is an arbitrary constant.

[7 marks]

- (c) Determine the general solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 8x^2$.

[8 marks]

Total 25 marks

END OF TEST

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SUBJECT: PURE MATHEMATICS – UNIT 2 – Paper 02

PROFICIENCY: ADVANCED

REGISTRATION NUMBER:

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Date: _____

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CARIBBEAN ADVANCED PROFICIENCY EXAMINATIONS®

PURE MATHEMATICS

UNIT 2 – Paper 02

KEY AND MARK SCHEME

MAY/JUNE 2022

SPECIMEN PAPER

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
1.	(a) (i)	$\frac{3-4i}{2+i} = \frac{(3-4i)(2-i)}{(2+i)(2-i)}$ [1 mark] $= \frac{6-8i-3i-4}{4+1}$ [1 mark] $\text{Simplification} = \frac{2-11i}{5}$ [1 mark] $\text{Numerator \& denominator correct} = \frac{2}{5} - \frac{11}{5}i$ [1 mark] "his" answer in the form a + ib	4
	(ii)	Principal argument: $\text{Arg } \theta = \tan^{-1} \frac{-11/5}{2/5}$ [1 mark] Correct form of argument $\theta = \tan^{-1} \frac{-11}{2} \text{ simplification}$ [1 mark] $\theta = -1.39 \text{ rad CAO}$ [1 mark]	3

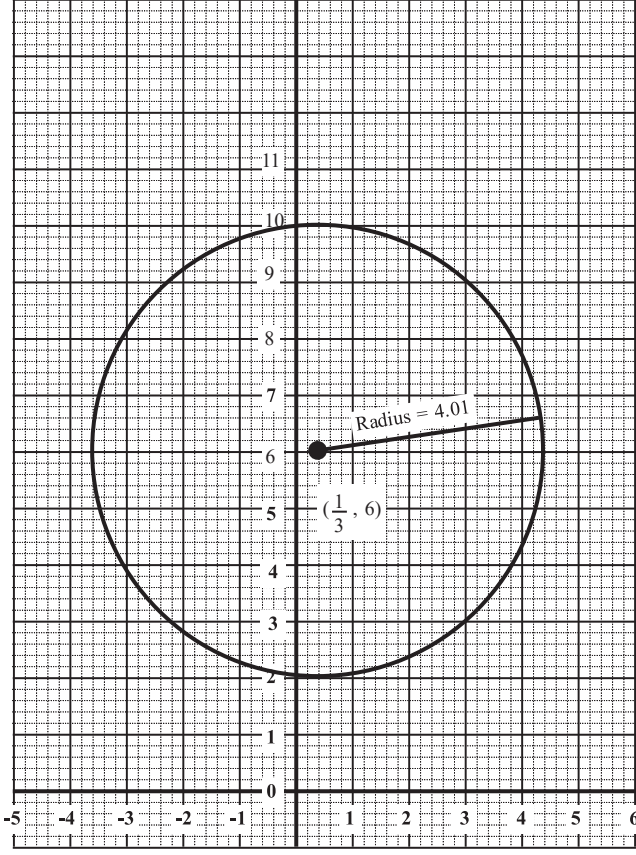
PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(b) (i)	Derive the complex number of the form $u = x + iy,$ where $x, y \in R$, and $u^2 = -3 + 4i$ $(x + iy)^2 = -3 + 4i$ $\Rightarrow x^2 + 2ixy - y^2 = -3 + 4i$ expansion of u^2 [1 mark] $\Rightarrow x^2 - y^2 = -3$ equating real and imaginary parts $2xy = 4$ [1 mark] $\Rightarrow xy = 2$ $\Rightarrow y = \frac{2}{x}$ Simplifying for y [1 mark] $\Rightarrow x^2 - \left(\frac{2}{x}\right)^2 = -3$ $\Rightarrow (x^2)^2 - 3x^2 - 4 = 0$ Simplifying equation [1 mark] $\Rightarrow (x^2 - 4)(x^2 + 1) = 0$ $\therefore x^2 - 4 = 0 \text{ or } x^2 = -1$ (inadmissible) solving $\Rightarrow x = \pm 2 \text{ and } y = \mp 2$ Required solution: $2 - 2i, -2 + 2i$ [1 mark] at least one solution [1 mark]	6

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(ii)	Hence solve the equation $z^2 - z + (1-i) = 0$ $z = \frac{-i \pm \sqrt{-3+4i}}{2} \quad [1 \text{ mark}]$ $z = \frac{-i \pm (2-2i)}{2} \quad \text{OR} \quad \frac{-i \pm (2-2i)}{2} \quad [1 \text{ mark}]$ correct values for $\sqrt{-3+4i}$ $z = \frac{2-3i}{2} \quad \text{OR} \quad z = \frac{-2+i}{2} \quad [1 \text{ mark}]$ correct simplification of 'his' values $z = 1 - \frac{3}{2}i \quad \text{OR} \quad z = -1 + \frac{1}{2}i \text{ in the form } a+bi \quad [1 \text{ mark}]$	4
	(c) (i)	The complex number $z=x+iy$ satisfies the equation $ z-3+2i =2 z-1-4i $ The complex number z is represented by the point P in an Argand diagram. Show that the locus of P is a circle. $ x+iy-3+2i =2 x+iy-1-4i $ $ (x-3)+(y+2)i =2 (x-1)+(y-4)i \quad [1 \text{ mark}]$ Substitution $(x-3)^2 + (y+2)^2 = 4(x-1)^2 + 4(y-4)^2$ Modulus $x^2 - 6x + 9 + y^2 + 4y + 4 = 4x^2 - 8x + 4 + 4y^2 - 32y + 64 \quad [1 \text{ mark}]$ $3x^2 + 3y^2 - 2x - 36y + 60 = 0 \quad [1 \text{ mark}]$ (1 mark) Simplification $x^2 + y^2 - \frac{2}{3}x - 12y + 20 = 0 \quad [1 \text{ mark}]$ Correct equation	4

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(c) (ii)	State the centre and radius of the circle. Centre = $\left(\frac{1}{3}, 6\right)$ [1 mark] Radius = 4.01 [1 mark]	2
	(iii)	 Drawing of circle [1 mark] Displaying the centre and radius [1 mark]	2
		TOTAL	25
		Specific Objectives: 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.8, 1.10	

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
2.	(a)	Let $I = \int e^{2x} \cos 3x \, dx$ $= \frac{1}{2} e^{2x} \cos 3x + \int \frac{1}{2} e^{2x} 3 \sin 3x \, dx$ Integration by parts correct process [1 mark] $= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx$ [1 mark] at least 1 part correct $= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \left[\frac{1}{2} e^{2x} \sin 3x - \int \frac{1}{2} e^{2x} 3 \cos 3x \, dx \right]$ [1 mark] Repeat integration $= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} \int e^{2x} \cos 3x \, dx$ $I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} I$ [2 marks] $\frac{13}{4} I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x$ $I = \frac{4}{13} \times \frac{1}{4} e^{2x} [2 \cos 3x + 3 \sin 3x]$ [1 mark] Simplification $I = \frac{1}{13} e^{2x} (2 \cos 3x + 3 \sin 3x) + c$ [1 mark] <u>Alternate Solution</u> $I = \operatorname{Re} \int e^{2x} e^{3ix} \, dx$ Real part $= \int e^{(2+3i)x} \, dx$ exponential form $\Rightarrow \operatorname{Re} \left[\frac{e^{(2+3i)x}}{2+3i} \right] + \text{constant}$ integration $= \operatorname{Re} \frac{2-3i}{(2+3i)(2+3i)} e^{2x} (\cos 3x + i \sin 3x)$ conjugation $= \operatorname{Re} \frac{e^{2x}}{13} (2-3i)(\cos 3x + i \sin 3x)$ simplification $\Rightarrow \int e^{2x} \cos 3x \, dx = \frac{e^{2x}}{13} (2 \cos 3x + 3 \sin 3x)$ real part $\frac{e^{2x}}{13} (3 \sin 2x - 2 \cos 2x) + \text{const.} + \text{constant}$	7

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(b) (i)	$x^2 + 4y^2 = 1$ $2x + 8y \frac{dy}{dx} = 0$ (1 mark) [1 mark] differentiating x^2 differentiating $4y^2$ $8y \frac{dy}{dx} = -2x$ (1 mark) [1 mark] $\frac{dy}{dx} = \frac{-x}{4y}$ (1 mark) [1 mark]	3
	(ii)	$\frac{d^2y}{dx^2} = \frac{-4y + 4x \frac{dy}{dx}}{16y^2}$ correct formula = $\frac{-y + x \frac{dy}{dx}}{4y^2}$ [1 mark] correct substitution = $\frac{-y - \frac{x^2}{4y}}{4y^2}$ [1 mark] $= \frac{-4y^2 - x^2}{16y^3}$ [1 mark] simplification = $\frac{-1}{16y^3}$ [1 mark]	4

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(c)	$u = 1 + x^2$ $du = 2x dx$ [1 mark] Differentiating When $x = 0, u = 1$ When $x = 1, u = 2$ [1 mark] $\int_0^1 \frac{x^3}{(1+x^2)^4} dx = \int_1^2 \frac{1}{2} \left(\frac{u-1}{u^4} \right) du$ $\text{substituting} = \frac{1}{2} \int_1^2 \left(\frac{1}{u^3} - \frac{1}{u^4} \right) du = \frac{1}{2} \int_1^2 (u^{-3} - u^{-4}) du$ [1 mark] $= \frac{1}{2} \left[\frac{-1}{2} u^{-2} + \frac{1}{3} u^{-3} \right]_1^2$ [1 mark] $\text{integration} = \frac{1}{2} \left[\frac{-1}{8} + \frac{1}{24} + \frac{1}{2} - \frac{1}{3} \right]$ [2 marks] $\text{evaluation} = \frac{1}{24}$ [1 mark]	7
	(d) (i)	$\frac{\partial f}{\partial x} = 6x - y$ [1 mark] $\frac{\partial f}{\partial y} = -x - 6y$ [1 mark]	2
	(ii)	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ [1 mark] Show that $6 - 6 = 0$ [1 mark]	2
		TOTAL	25
		Specific Objectives: 2.4, 2.7, 2.8, 3.6, 3.7	

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
3.	(a) (i)	$\Rightarrow \frac{1}{(1-2r)(1+2r)} = \frac{A}{1-2r} + \frac{B}{1+2r} \quad [1 \text{ mark}]$ $\Rightarrow 1 = A(1+2r) + B(1-2r) \quad [1 \text{ mark}]$ $\Rightarrow 0 = 2A - 2B \text{ and } A + B = 1 \quad [1 \text{ mark}]$ $A = \frac{1}{2} \text{ and } B = \frac{1}{2} \quad [2 \text{ marks}]$	5
	(ii)	$s = \sum_{r=1}^n \frac{1}{(1-2r)(1+2r)} = \sum_{r=1}^n \frac{1}{2} \left(\frac{1}{1-2r} + \frac{1}{1+2r} \right) \quad [1 \text{ mark}]$ $= \frac{1}{2} \left(-\frac{1}{1} + \frac{1}{3} \right) + \frac{1}{2} \left(-\frac{1}{3} + \frac{1}{5} \right) + \frac{1}{2} \left(-\frac{1}{5} + \frac{1}{7} \right) + \dots + \frac{1}{2} \left(-\frac{1}{1-2n} + \frac{1}{1+2n} \right) \quad [3 \text{ marks}]$ Any three brackets, 1 mark each $= \frac{1}{2} \left(-1 + \frac{1}{1+2n} \right) \quad [1 \text{ mark}]$	5
	(iii)	As $n \rightarrow \infty$, $\frac{1}{1+2n} \rightarrow 0$ [2 marks] Hence $S_{\infty} = -\frac{1}{2}$ [1 mark]	3
	(b) (i)	$S = 1(8) + 2(20) + 3(32) + \dots$ [1 mark] In each term, 1st factor is in the natural sequence and the second factor differs by 12 $\Rightarrow$ the r^{th} term is $r(12r-4)$ [1 mark]	2

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(ii)	$S_n = \sum_{r=1}^n r(12r-4) \text{ for } n=1$ $S_1 = \sum_{r=1}^1 r(12r-4) = 1 \times 8 = 8$ and $2(1)^2 [2(1)+2] = 2 \times 4 = 8$ hence, $S_n = 2n^2(2n+2)$ [1 mark] is true for $n=1$ [1 mark] Assume $S_n = 2n^2(2n+2)$ for $n=k \in \mathbb{N}$ [1 mark] that is, $S_k = 2k^2(2k+2)$ [1 mark] Then, $S_{k+1} = \sum_{r=1}^{k+1} (12r-4) = S_k + (k+1)(12k+8)$ [1 mark] $= 2k^2(2k+2) + (k+1)(12k+8)$ [1 mark] $= 4k^2(k+1) + 4(k+1)(3k+2)$ [1 mark] $= 4(k+1)[k^2+3k+2]$ [1 mark] $\Rightarrow S_{k+1} = 4(k+1)[(k+1)+(k+2)]$ $= 4(k+1)^2(k+2)$ [1 mark] $= 2(k+1)^2 \times 2[(k+1)+1]$ $= 2(k+1)^2 [2(k+1)+2]$ [1 mark] $\Rightarrow$ true for $n=k+1$ whenever it is assumed true for $n=k$, $\Rightarrow$ true for all $n \in \mathbb{N}$ $\Rightarrow S_n = 2n^2(2n+2) \quad n \in \mathbb{N}$ [1 mark]	10
		TOTAL	25
		Specific Objectives: 2.2, 2.4, 2.5, 2.6, 3.1	

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
4.	(a) (i)	$s = \frac{1}{3} + \frac{1}{3^4} + \frac{1}{3^7} + \frac{1}{3^{10}} + \dots$ $\frac{\frac{1}{3^4}}{\frac{1}{3}} = \frac{\frac{1}{3^7}}{\frac{1}{3^4}}$ [1 mark] Let $u = \frac{1}{3^3}$ [1 mark] $\therefore S$ is geometric with common ratio [1 mark] $r = \frac{1}{3^3} \text{ or } \frac{1}{27}$	3
	(ii)	$S_n = \frac{\frac{1}{3} \left[1 - \left(\frac{1}{3} \right)^{3n} \right]}{1 - \left(\frac{1}{3} \right)^3}$ [1 mark] $= \frac{\frac{1}{3} \left[1 - \frac{1}{3^{3n}} \right]}{1 - \frac{1}{27}}$ [1 mark] $= \frac{1}{3} \times \frac{27}{26} \left[1 - \frac{1}{3^{3n}} \right]$ [1 mark] $= \frac{9}{26} \left[1 - \frac{1}{3^{3n}} \right] \text{ or } \frac{9}{26} \left[1 - \frac{1}{27^n} \right]$ [1 mark]	4

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(b)	$f(x) = \sin 2x$ $\Rightarrow f'(x) = 2 \cos 2x$ [1 mark] $\Rightarrow f''(x) = -4 \sin 2x$ [1 mark] $\Rightarrow f'''(x) = -8 \cos 2x$ [1 mark] $\Rightarrow f^{iv}(x) = 16 \sin 2x$ [1 mark] so, $f(0) = 0, f'(0) = 2, f''(0) = 0, f'''(0) = -8, f^{iv}(0) = 0$ [1 mark] Hence, by Maclaurin's Theorem, $\sin 2x = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} + \dots$ [1 mark] $= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5$ [1 mark]	7
	(c) (i)	$\sqrt{\frac{1+x}{1-x}}$ $= (1+x)^{1/2} (1-x)^{-1/2}$ [1 mark] $= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \dots\right) \left(1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 \dots\right)$ [3 marks] $= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$ [2 marks] for $-1 < x < 1$ [1 mark]	7
	(ii)	$\sqrt{\frac{1.04}{0.96}} = \sqrt{\frac{104}{96}} = \sqrt{\frac{13}{12}}$ [1 mark] $\sqrt{\frac{52}{3}} = \sqrt{\frac{1+x}{1-x}}$ [1 mark] Where $x = 0.04$ $\Rightarrow \sqrt{\frac{52}{3}} = 4 \left\{ 1 + 0.04 + \frac{1}{2}(0.04)^2 + \frac{1}{2}(0.04)^3 \right\}$ [1 mark] $= 4.163$ (3 d.p.) [1 mark]	4
		TOTAL	25
		Specific Objectives: 2.2, 2.3, 2.7, 3.2	

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
5.	(a) (i)	2 fair coins and 1 fair die are tossed at the same time. The number of outcomes for the coins are HH, HT, TH, TT so we have 4 outcomes and the number of outcomes for the die is 6. Therefore, there are $4 \times 6 = 4$ outcomes. Number of outcomes on the die [1 mark] Number of outcomes on the 2 coins [1 mark] Computation of all possible outcomes [1 mark]	3
	(ii)	There are 2 outcomes with 1 head. Therefore, we have $2 \times 6 = 12$ outcomes with exactly 1 head. The probability of obtaining exactly 1 head is given by $\frac{12}{24} = 0.5$. Recognition that this can be obtained 2 possible ways (HT, TH) with any of the 6 possibilities on the die [1 mark] Computation of probability based on the space [1 mark]	2

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(iii)	The number of outcomes with at least 1 head is 3 and the number of even numbers is 3. Therefore there are $3 \times 3 = 9$ outcomes with at least one head and an even number. The probability of obtaining at least 1 head and an even number is given by $\frac{9}{24} = 0.375$ Recognition that there are 3 cases with at least 1 head. [1 mark] Recognition of number of outcomes on the die [1 mark] Calculation of the number of possibilities [1 mark] Calculation of probability [1 mark]	4
	(b) (i)	The probability that it is non-fiction, borrowed by a boy is given by $\frac{5}{100} = \frac{1}{20}$ Recognition that the cell intersecting non-fiction and boy is 5 [1 mark] Total possible outcomes = 100 [1 mark]	2
	(ii)	The probability that it is fiction, borrowed by a female is given by $\frac{30}{100} = \frac{3}{10}$ Recognition that the cells women and girl intersecting fiction total 30 [1 mark] Total possible outcomes = 100 [1 mark] Correct computation [1 mark]	3

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(iii)	The probability that it was borrowed by a man(M), given that it is fiction(F) is given by $P(M / F) = \frac{P(M \cap F)}{P(F)} = \frac{17/100}{61/100} = \frac{17}{61}$ Recognition that it is conditional probability [1 mark] Correct formula used [1 mark] Correct answer [1 mark]	3
	(c) (i)	Let T, S and F represent respectively the customers purchasing tools, seeds and fertilizer. One mark for each circle correct [3 marks] Venn diagram	3

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(ii) a)	$P(T \cup S)' = \frac{39}{100}$ Numerator (19 + 20 = 39) [1 mark] Correct answer $\frac{39}{100}$ [1 mark]	2
	b)	$P(S) = \frac{40}{100} \text{ (from candidate's diagram)}$ $P(F) = \frac{56}{100} \text{ (from candidate's diagram)}$ $P(S \cap F) = \frac{30}{100} \text{ (from candidate's diagram)}$ $P(S) \times P(F) = \frac{28}{125}$ Since $P(S \cap F) \neq P(S) \times P(F)$ then S and F are not independent $P(S \cap F) = \frac{30}{100} \quad [1 \text{ mark}]$ $P(S) \times P(F) = \frac{28}{125} \quad [1 \text{ mark}]$ Conclusion (not independent or dependent) [1 mark]	3
		TOTAL	25
		Specific Objectives: 1.4, 1.5, 1.6, 1.7, 1.9, 1.12, 1.13	

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
6.	(a) (i)	Given $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -1 & 6 & 0 \end{pmatrix}$ $A = 2 \begin{vmatrix} 4 & 3 \\ 6 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 4 \\ -1 & 6 \end{vmatrix}$ [3 marks] $A = 2(-18) - 1(3) - (4) = -43$ [1 mark] Determinant of the A [1 mark]	5
	(ii)	Cofactor of $A = \begin{pmatrix} -18 & -3 & 4 \\ -6 & -1 & -13 \\ 7 & -6 & 8 \end{pmatrix}$ Adjoint of $A = \begin{pmatrix} -18 & -6 & 7 \\ -3 & -1 & -6 \\ 4 & -13 & 8 \end{pmatrix}$ $A^{-1} = -\frac{1}{43} \begin{pmatrix} -18 & -6 & 7 \\ -3 & -1 & -6 \\ 4 & -13 & 8 \end{pmatrix}$ One mark for each correct row [3 marks] Transposing the matrix of cofactors [1 mark] Multiplying the reciprocal of the determinant by the Adjoint matrix [1 mark]	5

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(b)	$e^{\int \frac{1}{t+10} dt} = e^{2\ln(t+10)} = (t+10)^2$ $y(t+10)^2 = 3 \int (t+10)^2$ $= 3 \frac{(t+10)^3}{3} + c$ $= (t+10)^3 + c$ $y = \frac{(t+10)^3 + c}{(t+10)^2}$ $= t+10 + \frac{c}{(t+10)^2}$ Integrating $\frac{1}{t+10}$ correctly [1 mark] Integrating factor ([1 mark] Multiplying the DE by the IF [1 mark] Integrating [1 mark] Dividing by IF [1 mark] Isolating y [1 mark] Expressing the answer in the correct form [1 mark]	7

PURE MATHEMATICS
UNIT 2 – Paper 02
KEY AND MARK SCHEME

	Question	Solutions	Total
	(c)	$m^2 - 3m - 4 = 0$ $(m - 4)(m + 1) = 0$ $m = 4; m = -1$ $y_c = Ae^{4x} + Be^{-x}$ $y_p = Cx^2 + Dx + E$ $y'_p = 2Cx + D$ $y''_p = 2C$ Sub into DE $2C - 3(2Cx + D) - 4(Cx^2 + Dx + E) = 8x^2$ $2C - 6Cx - 3D - 4Cx^2 - 4Dx - 4E = 8x^2$ Equating coefficients $-4C = 8 \Rightarrow C = -2$ $-6C - 4D = 0$ $-6(-2) - 4D = 0 \Rightarrow D = 3$ $2C - 3D - 4E = 0$ $2(-2) - 3(3) - 4E = 0 \Rightarrow E = -\frac{13}{4}$ $\therefore y = Ae^{4x} + Be^{-x} - 2x^2 + 3x - \frac{13}{4}$ Auxiliary equation [1 mark] Complementary function [1 mark] Particular integral [1 mark] Derivative [1 mark] Values of C, D, E [3 marks] General solution [1 mark]	8
		TOTAL	25
		Specific Objectives: 2.3, 2.4, 3.1, 3.2	

SPECIMEN 2022



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PURE MATHEMATICS

UNIT 2

COMPLEX NUMBERS, ANALYSIS AND MATRICES

SPECIMEN PAPER

PAPER 032

2 hours

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This examination paper consists of THREE questions.
2. Answer ALL questions.
3. Write your answers in the spaces provided in this booklet.
4. Do NOT write in the margins.
5. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.
6. If you need to rewrite any answer and there is not enough space to do so on the original page, you must use the extra page(s) provided at the back of this booklet. **Remember to draw a line through your original answer.**
7. **If you use the extra page(s) you MUST write the question number clearly in the box provided at the top of the extra page(s) and, where relevant, include the question part beside the answer.**

***The questions on this paper may be based on Specific Objectives taken from ANY Module in the Unit.**

Examination Materials Permitted

Mathematical formulae and tables (provided) – **Revised 2022**

Mathematical instruments

Silent, non-programmable electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

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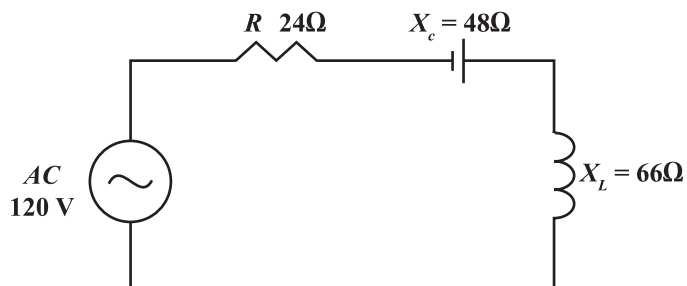
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SECTION A

Module 1

1. The diagram below shows an electrical circuit consisting of a resistor (R), an inductor (L) and a capacitor (C).



- (a) The impedance, Z , of the circuit may be represented by the equation $Z = R + i|X_L - X_C|$.

With the aid of a suitable diagram, express the impedance in polar form where $-\pi < \arg Z \leq \pi$.

[7 marks]

GO ON TO THE NEXT PAGE

- (b) Sugar is poured into a cylindrical tower. As the sugar is poured it forms a right circular cone. The base radius is increasing at the rate of 3m/s, while the perpendicular height of the cone is increasing at a rate of 1.2m/s.

[Volume of a right circular cone, $V = \frac{1}{3}\pi r^2 h$]

- (i) Write an expression for the rate of change of the volume with respect to

a) the radius

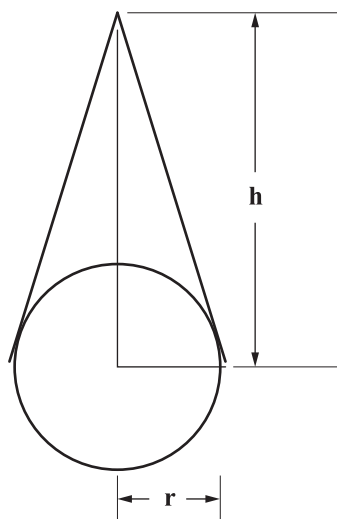
[2 marks]

b) the height

[1 mark]

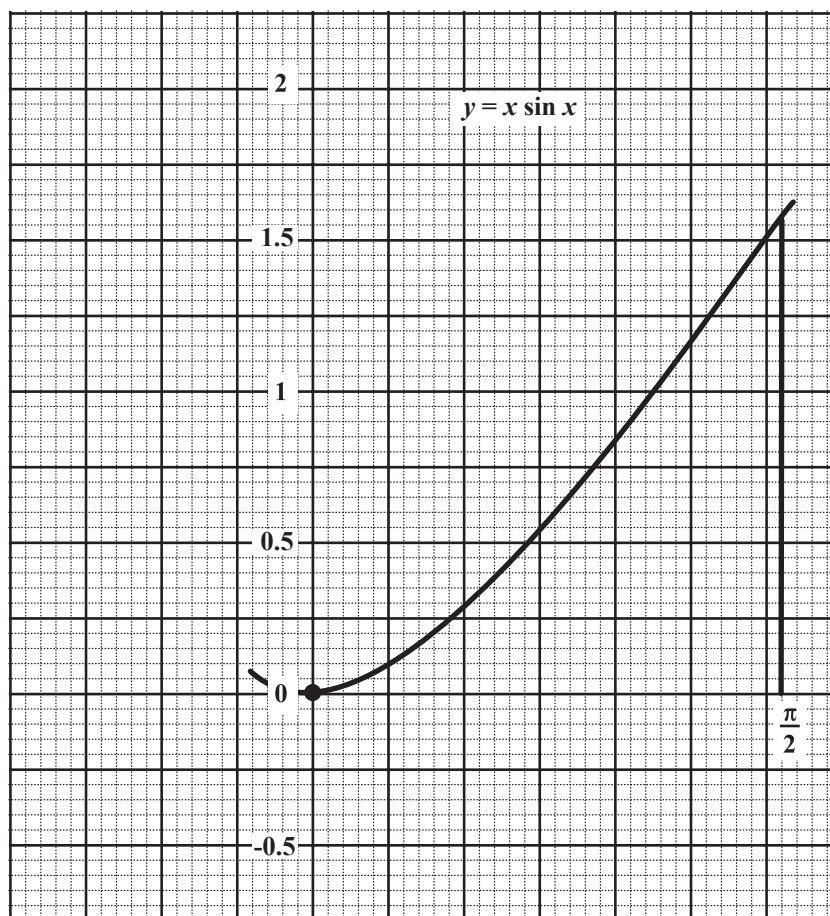
- (ii) Given that $r = 2.4m$ and $h = 4.8m$, determine the rate at which the volume, V , is changing.

Hint: $\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$



[3 marks]

- (c) The diagram below, **not drawn to scale**, shows the curve $y = x \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.



- (i) Determine the exact value of $\int_0^{\frac{\pi}{2}} x \sin x \, dx$.

[3 marks]

- (ii) Using the trapezium rule, with 2 intervals, estimate the value of $\int_0^{\frac{\pi}{2}} x \sin x \, dx$, giving your answer correct to 2 decimal places.

[2 marks]

- (iii) With the aid of a geometrical representation, explain why the trapezium rule gives an overestimate of the value found in (c) (i).

[2 marks]

Total 20 marks

GO ON TO THE NEXT PAGE

SECTION B

Module 2

2. (a) In a certain country, a construction company agrees to pay its construction workers a starting salary of \$2000 per month in the first year of employment. Workers will then be entitled to a yearly increase of 5% of the current years' monthly salary.

- (i) How many years will it take Mr Jack to receive a salary of \$2555 per month?

[3 marks]

- (ii) What will be the TOTAL salary paid to a person for the first three years of working with the company.

[4 marks]

- (iii) No increases in salaries are given after 10 years working with the company. Calculate the monthly salary of a person who has been with the company for more than 10 years.

[2 marks]

GO ON TO THE NEXT PAGE

- (b) (i) A function is given by the formula $f(x) = 1 - 2x$ for all positive values of x . Generate the expansion of $(1 - 2x)^5$ giving each term in its simplest form.

[3 marks]

- (ii) If x is small enough so that x^2 and higher powers of x can be ignored, show that $(1 + x)(1 - 2x)^5 \approx 1 - 9x$.

[2 marks]

- (c) (i) Show that the function $f(x) = x^3 + x - 3$ has a root between $x = 0$ and $x = 4$.

[2 marks]

- (ii) Use the method of interval bisection to estimate the value of the root correct to 1 decimal place.

[4 marks]

Total 20 marks

GO ON TO THE NEXT PAGE

SECTION C

Module 3

3. (a) At a fruit store, Mary paid \$30 for 2 kg of apples, 1 kg of cherries and 4 kg of plums. Gerry paid \$38 for 1 kg of apples, 2 kg of cherries and 1 kg of plums. Lester paid \$42 for 1 kg of apples, 1 kg of cherries and 2 kg of plums.

- (i) Write this information as a matrix equation in the form $AX = B$.

[3 marks]

- (ii) Using the row reduction method, calculate the cost per kilogram of each of the three fruits.

[5 marks]

GO ON TO THE NEXT PAGE

(b) A committee of 8 persons is to be seated at a round table.

(i) In how many ways can this be done

a) without restrictions

[1 mark]

b) if the chairman and the secretary must sit next to each other.

[2 marks]

(ii) Calculate the probability that the chairman and the secretary are NOT seated next to each other.

[2 marks]

GO ON TO THE NEXT PAGE

- (c) (i) The amount of salt, y kg, that dissolves in a tank of water after a time t minutes satisfies the first order differential equation $\frac{dy}{dt} + 2y(t+10) = 3$. Using a suitable integrating factor, show that the general solution of this differential equation is $y = t + 10 + \frac{c}{(t+10)^2}$ where c is an arbitrary constant.

[5 marks]

- (ii) Initially the amount of salt put into the tank is 5kg. How much salt will be dissolved after 15 minutes?

[2 marks]

Total 20 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

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INSTRUCTIONS TO CANDIDATE:

1. Fill in all the information requested clearly in capital letters.

TEST CODE:

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SUBJECT: PURE MATHEMATICS – UNIT 2 – Paper 032

PROFICIENCY: ADVANCED

REGISTRATION NUMBER:

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FULL NAME: _____
(BLOCK LETTERS)

Signature: _____

Date: _____

2. Ensure that this slip is detached by the Supervisor or Invigilator and given to you when you hand in this booklet.
3. Keep it in a safe place until you have received your results.

INSTRUCTION TO SUPERVISOR/INVIGILATOR:

Sign the declaration below, detach this slip and hand it to the candidate as his/her receipt for this booklet collected by you.

I hereby acknowledge receipt of the candidate's booklet for the examination stated above.

Signature: _____
Supervisor/Invigilator

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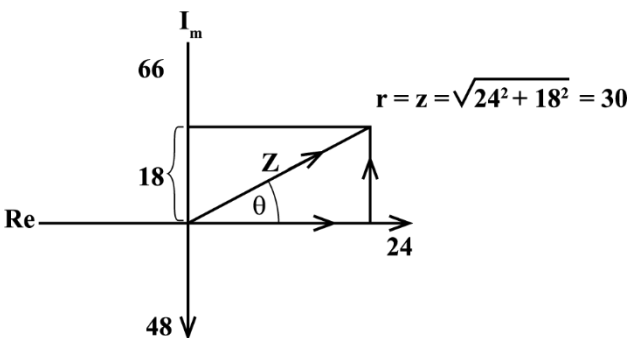
UNIT 2 – Paper 032

KEY AND MARK SCHEME

MAY/JUNE 2022

SPECIMEN PAPER

PURE MATHEMATICS
UNIT 2 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total
1.	(a)	The impedance is: $Z = R + i X_L - X_C$ $z = \sqrt{R^2 + X_L - X_C ^2}$ [1 mark] $= \sqrt{24^2 + 66 - 48 ^2}$ [1 mark] $= \sqrt{576 + 324}$ [1 mark] $= 30$ [1 mark] $\arg Z = \tan^{-1}\left(\frac{18}{24}\right)$ [1 mark] $= 0.643 \text{ radians}$ The polar form of the impedance, $Z = 30(\cos 0.643 + i \sin 0.643)$ [1 mark]  [1 mark]	7
	(b) (i)	$\frac{\partial V}{\partial r} = \frac{2\pi rh}{3}$ [2 marks] $\frac{\partial V}{\partial r} = \frac{\pi r^2}{3}$ [1 mark]	3

PURE MATHEMATICS
UNIT 2 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total								
	(ii)	Given that $r = 2.4m$, $h = 4.8m$, then, $\frac{dr}{dt} = 3ms^{-1}$ and $\frac{dh}{dt} = 1.2ms^{-1}$ [1 mark] $\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$ $\frac{dV}{dt} = \frac{2\pi(2.4)(4.8)}{3} \times 3 + \frac{\pi(2.4)^2}{3} \times (-1.2)$ [1 mark] $= 23.04\pi - 23.04\pi$ $= 46.08\pi \text{ m}^3 \text{ s}^{-1}$ [1 mark]	3								
	(c) (i)	Let $v = x$ $\frac{du}{dx} = \sin x$ [1 mark] $\frac{dv}{dx} = 1$ $u = -\cos x$ $\int_0^{\frac{\pi}{2}} x \sin x \, dx = [-x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$ [1 mark] $= [\sin x]_0^{\frac{\pi}{2}}$ $= \sin\left(\frac{\pi}{2}\right) - \sin 0$ $= 1$ [1 mark]	3								
	(ii)	<table border="0"> <tr> <td>x</td> <td>0</td> <td>$\frac{\pi}{4}$</td> <td>$\frac{\pi}{2}$</td> </tr> <tr> <td>y</td> <td>0</td> <td>0.5554</td> <td>1.5708</td> </tr> </table> $A = \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right)[(0 + 1.5708) + 2(0.5554)]$ [1 mark] $= 1.05306$ $= 1.05(2 \text{ d.p.})$ [1 mark]	x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	y	0	0.5554	1.5708	2
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$								
y	0	0.5554	1.5708								

PURE MATHEMATICS
UNIT 2 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total
	(iii)	$y = x \sin x$ [1 mark] Since the trapezia lies above the curve, the trapezium rule is an over-estimate of the true value. [1 mark]	2
		TOTAL	20
		Specific Objectives: Module 1, 1.6; 1.7; 1.8, 1.12, 2.8, 3.1, 3.2, 3.7, 3.11	

PURE MATHEMATICS
UNIT 2 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total
2.	(a) (i)	$t_n = t_a r^{n-1}$ formula for the nth term of a sequence $t_n = 2550 \quad t_a = 2000 \quad r = 1.05$ [1 mark] $2555 = 2000(1.05)^{n-1}$ correct substitution into formula $\frac{2555}{2000} = 1.05^{n-1}$ $\frac{\log 1.2775}{\log 1.05} = n - 1$ [1 mark] Correct application of logarithms $n = 6.01$ years [1 mark]	3
	(ii)	Since salaries are paid month, it will be necessary to calculate the sum per year, and then add. Year 1: $12 \times 2000 = 24000$ [1 mark] Year 2: $12 \times 2100 = 25200$ [1 mark] Year 3: $12 \times 2205 = 26460$ [1 mark] Total salary for 3 years = \$75660 [1 mark] addition of their values	4
	(iii)	$t_{10} = 2000(1.05)^9$ substitution into formula = 3102.66 [1 mark] Monthly salary after 10 years will be \$3102.66 [1 mark]	2
	(b) (i)	Generate the expansion of $(1-2x)^5$ giving each term in its simplest form. $1 + 5(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + {}^5C_4(-2x)^4 + {}^5C_5(-2x)^5$ Expansion with 4 to 6 terms correct [2 marks] [3 terms correct, 1 mark] $= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$ [1 mark] simplifying terms	3

PURE MATHEMATICS
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KEY AND MARK SCHEME

	Question	Solutions	Total
	(ii)	$(1+x)(1-2x)^5$ $=1-10x+40x^2+\dots+x-10x^2+\dots$ [1 mark] Multiplying the brackets $=1-10x+x+\dots$ [1 mark] Choosing only correct terms $\approx 1-9x$	2
	(c) (i)	$f(0)=-3$ $f(4)=65$ [1 mark] the two signs are different, therefore the functions cross the x-axis between 0 and [1 mark]	2
	(ii)	mid-point of $[0,4]$ is $x=2$ [1 mark] $f(2)=7$ root lies between $x=0$ and $x=2$ [1 mark] midpoint of $[0,2]$ is $x=1$ $f(1)=-1$ roots lie between $x=1$ and $x=2$ midpoint of $[1,2]$ is $x=1.5$ $f(1.5)=1.83$ root lies between $x=1$ and $x=1.5$ midpoint of $[1, 1.5]$ is 1.25 $f(1.25)=1.32$ [1 mark] root lies between $x=1$ and $x=1.25$ midpoint of $[1, 1.25]$ is 1.125 $f(1.125)=1.141$ root can be approximated $x=1$ [1 mark]	4
		TOTAL	20
		Specific Objectives: Module 2, 2.2, 3.2, 4.1, 4.2	

PURE MATHEMATICS
UNIT 2 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total
3.	(a) (i)	$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 30 \\ 38 \\ 42 \end{pmatrix}$ x is the cost of 1 kg apples [1 mark] y is the cost of 1 kg cherries z is the cost of 1 kg plums variables defined [1 mark] matrix A correct [1 mark] matrix B correct	3
	(ii)	Form an augmented matrix and use the method of row reduction to reduce the Augmented matrix to echelon form. $\left(\begin{array}{ccc c} 2 & 1 & 1 & 30 \\ 1 & 2 & 1 & 38 \\ 1 & 1 & 2 & 42 \end{array} \right)$ Augmented matrix [1 mark] $\left(\begin{array}{ccc c} 2 & 1 & 1 & 30 \\ 0 & 3 & 1 & 46 \\ 0 & 1 & 3 & 54 \end{array} \right)$ 2R₂–R₁ 2R₃–R₁ $\left(\begin{array}{ccc c} 2 & 1 & 1 & 30 \\ 0 & 3 & 1 & 46 \\ 0 & 0 & 8 & 116 \end{array} \right)$ 2R₃–R₃	1

PURE MATHEMATICS
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KEY AND MARK SCHEME

	Question	Solutions	Total
		Row reduction to echelon form [2 marks] $8z = 116$ $z = \frac{116}{8} = 14.50$ $3y + z = 46$ $3y = 46 - z$ $y = \frac{46 - 14.5}{3}$ $= 10.50$ $2x + y + z = 30$ $2x = 30 - y - z$ $x = \frac{30 - 10.5 - 14.50}{2}$ $= 2.50$ $x = \$2.50; y = \$10.50; z = \$14.50$ at least 2 variables correct [2 marks]	4
	(b) (i)	In how many ways can the committee be seated a. With no restrictions $7! = 5040$ [1 mark] b. The chairman and the secretary must sit together. $6! \times 2 = 1440$ [2 marks]	3

PURE MATHEMATICS
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KEY AND MARK SCHEME

	Question	Solutions	Total
	(ii)	Probability that the chairman and the secretary are NOT seated together $1 - \frac{6! \times 2}{7!}$ OR $1 - \frac{1440}{5040} = 1 - 0.286$ $= 1 - \frac{2}{7} = \frac{5}{7}$ Probability [1 mark] = 0.714 Subtraction from 1 [1 mark]	2
	(c) (i)	Integrating Factor = $e^{\int \frac{2}{t+10} dt}$ [1 mark] $= e^{2 \ln(t+10)}$ [1 mark] $= (t+10)^2 = 3 \int (t+10)^2 dt$ $= 3 \frac{(t+10)^3}{3} + c$ [2 marks] $= (t+10)^3 + c$ $y = \frac{(t+10)^3 + c}{(t+10)^2}$ [1 mark] $= t+10 + \frac{c}{(t+10)^2}$	5

PURE MATHEMATICS
UNIT 2 – Paper 032
KEY AND MARK SCHEME

	Question	Solutions	Total
	(ii)	At $t = 0$, $y = 5kg$, $5 = 10 + \frac{c}{(0+10)^2}$ $-5 = \frac{c}{100}$ $-500 = c \quad [1 \text{ mark}]$ $y = 25 - \frac{500}{625}$ $y = 25 - 0.8$ $= 24.2kg \quad [1 \text{ mark}]$	2
		TOTAL	20
		Specific Objectives: Module 3, 1.1, 1.2, 1.11, 2.5, 2.6, 2.9, 3.1	



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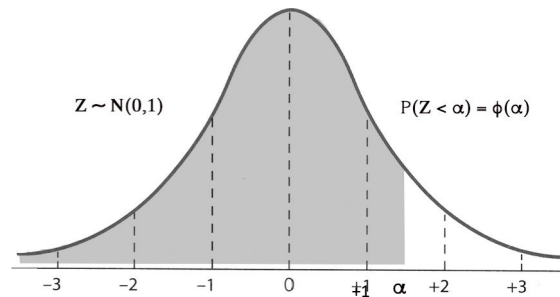
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Table 1: The Normal Distribution Function

If Z is a random variable, normally distributed with zero mean and unit variance, then $\phi(z)$ is the probability that $Z \leq z$. That is, $\phi(z) = P(Z \leq z)$.

The function tabulated below is $\phi(z)$, and is shown diagrammatically as

Standard Normal Distribution (area to the left of α)

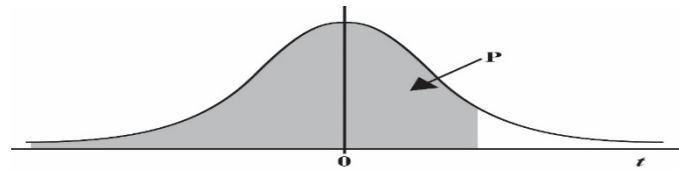


The Distribution Function, $\phi(z)$

Z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											ADD								
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	1	2	4	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	1	2	3	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	1	2	3	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	1	1	2	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9760	0.9767	1	1	2	1	1	2	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	0	1	1	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	0	1	1	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	0	1	1	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	0	1	1	2	2	2
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	0	0	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Table 2: t-Distribution

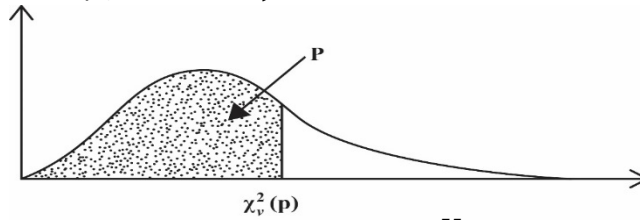
If T has a t -distribution with ν degrees of freedom then, for each pair of values of p and ν , the table gives the value of t such that $P(T \leq t) = P$

**Critical Values for the t-distribution**

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
$\nu = 1$	1.000	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	3.517	4.078	4.515
12	0.695	1.356	1.782	2.179	2.681	3.055	3.459	4.017	4.454
13	0.694	1.350	1.771	2.160	2.650	3.012	3.409	3.966	4.398
14	0.692	1.345	1.761	2.145	2.624	2.977	3.368	3.921	4.348
15	0.691	1.341	1.753	2.131	2.602	2.947	3.336	3.881	4.299
16	0.690	1.337	1.746	2.120	2.583	2.921	3.305	3.841	4.255
17	0.689	1.333	1.740	2.110	2.567	2.898	3.275	3.801	4.214
18	0.688	1.330	1.734	2.101	2.552	2.878	3.246	3.762	4.174
19	0.688	1.328	1.729	2.093	2.539	2.861	3.218	3.724	4.136
20	0.687	1.325	1.725	2.086	2.528	2.845	3.191	3.687	4.099
21	0.686	1.323	1.721	2.080	2.518	2.831	3.165	3.651	4.063
22	0.686	1.321	1.717	2.074	2.508	2.819	3.140	3.616	4.028
23	0.685	1.319	1.714	2.069	2.500	2.807	3.116	3.581	3.994
24	0.685	1.318	1.711	2.064	2.492	2.797	3.092	3.547	3.960
25	0.684	1.316	1.708	2.060	2.485	2.787	3.069	3.513	3.927
26	0.684	1.315	1.706	2.056	2.479	2.779	3.047	3.480	3.894
27	0.684	1.314	1.703	2.052	2.473	2.771	3.025	3.447	3.861
28	0.683	1.313	1.701	2.048	2.467	2.763	3.004	3.415	3.829
29	0.683	1.311	1.699	2.045	2.462	2.756	2.983	3.383	3.797
30	0.683	1.310	1.697	2.042	2.457	2.750	2.963	3.351	3.765
40	0.681	1.303	1.684	2.021	2.423	2.704	2.911	3.297	3.684
60	0.679	1.296	1.671	2.000	2.390	2.660	2.875	3.243	3.618
120	0.677	1.289	1.658	1.980	2.358	2.617	2.839	3.189	3.558
∞	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.499

Table 3: Percentage Points of the χ^2 Distribution

If X is a random variable, distributed as χ^2 with ν degrees of freedom then p is the probability that $X \leq \chi^2_\nu(p)$, where the values of the percentage points $\chi^2_\nu(p)$, are tabulated in the table below. p is shown diagrammatically (when $\nu \geq 3$) as

**Critical Values for the χ^2 -distribution**

P	.01	.025	.050	.900	.950	.975	.990	.995	.999
$\nu = 1$	0.0001571	0.0009821	0.003932	2.706	3.841	5.024	6.635	7.879	10.83
2	0.02010	0.05064	0.1026	4.605	5.991	7.378	9.210	10.60	13.82
3	0.1148	0.2158	0.3518	6.251	7.815	9.348	11.34	12.84	16.27
4	0.2971	0.4844	0.7107	7.779	9.488	11.14	13.28	14.86	18.47
5	0.5543	0.8312	1.145	9.236	11.07	12.83	15.09	16.75	20.52
6	0.8721	1.237	1.635	10.64	12.59	14.45	16.81	18.55	22.46
7	1.239	1.690	2.167	12.02	14.07	16.01	18.48	20.28	24.32
8	1.646	2.180	2.733	13.36	15.51	17.53	20.09	21.95	26.12
9	2.088	2.700	3.325	14.68	16.92	19.02	21.67	23.59	27.88
10	2.558	3.247	3.940	15.99	18.31	20.48	23.21	25.19	29.59
11	3.053	3.816	4.575	17.28	19.68	21.92	24.72	26.76	31.26
12	3.571	4.404	5.226	18.55	21.03	23.34	26.22	28.30	32.91
13	4.107	5.009	5.892	19.81	22.36	24.74	27.69	29.82	34.53
14	4.660	5.629	6.571	21.06	23.68	26.12	29.14	31.32	36.12
15	5.229	6.262	7.261	22.31	25.00	27.49	30.58	32.80	37.70
16	5.812	6.908	7.962	23.54	26.30	28.85	32.00	34.27	39.25
17	6.408	7.564	8.672	24.77	27.59	30.19	33.41	35.72	40.79
18	7.015	8.231	9.390	25.99	28.87	31.53	34.81	37.16	42.31
19	7.633	8.907	10.12	27.20	30.14	32.85	36.19	38.58	43.82
20	8.260	9.591	10.85	28.41	31.41	34.17	37.57	40.00	45.31
21	8.897	10.28	11.59	29.62	32.67	35.48	38.93	41.40	46.80
22	9.542	10.98	12.34	30.81	33.92	36.78	40.29	42.80	48.27
23	10.20	11.69	13.09	32.01	35.17	38.08	41.64	44.18	49.73
24	10.86	12.40	13.85	33.20	36.42	39.36	42.98	45.56	51.18
25	11.52	13.12	14.61	34.38	37.65	40.65	44.31	46.93	52.62
30	14.95	16.79	18.49	40.26	43.77	46.98	50.89	53.67	59.70
40	22.16	24.43	26.51	51.81	55.76	59.34	63.69	66.77	73.40
50	29.71	32.36	34.76	63.17	67.50	71.42	76.15	79.49	86.66
60	37.48	40.48	43.19	74.40	79.08	83.30	88.38	91.95	99.61
70	45.44	48.76	51.74	85.53	90.53	95.02	104.4	104.2	112.3
80	53.54	57.15	60.39	96.58	101.9	106.6	112.3	116.3	124.8
90	61.76	65.65	69.13	107.6	113.1	118.1	124.1	128.3	137.2
100	70.06	74.22	77.93	118.5	124.3	129.6	135.8	140.2	149.4

Table 4: Random Sampling Numbers

18	11	36	26	88	81	11	33	64	08	23	32	00	73	04
57	33	88	37	26	10	79	91	36	03	07	52	55	84	61
72	02	11	44	25	45	92	12	82	94	35	35	91	65	78
89	83	98	71	74	22	05	29	17	37	45	65	35	54	44
44	88	03	81	30	61	00	63	42	46	22	89	41	54	47
68	60	92	99	60	97	53	55	34	01	43	40	77	90	19
87	63	49	22	47	21	76	13	39	25	89	91	38	25	19
44	33	11	36	72	21	40	90	76	95	10	14	86	03	17
60	30	10	46	44	34	19	56	00	83	20	53	53	65	29
03	47	55	23	26	90	02	12	02	62	51	52	70	68	13
09	24	34	42	00	68	72	10	71	37	30	72	97	57	56
09	29	82	76	50	97	95	53	50	18	40	89	40	83	29
52	23	08	25	21	22	53	26	15	87	93	73	25	95	70
43	78	19	88	85	56	67	56	67	16	68	26	95	99	64
45 0	69	72	62	11	12	18	25	00	92	26	82	64	3	
21	72	97	04	52	62	09	54	35	17	22	73	35	72	53
65	95	48	55	12	46	89	95	61	31	77	14	24	14	41
51	69	76	00	20	92	58	21	24	33	74	08	66	90	61
89	56	83	39	58	22	09	01	14	04	14	97	56	92	97
72	63	40	03	07	02	62	20	11	50	11	98	23	80	99

FORMULAE

PURE MATHEMATICS

For the quadratic equation: $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n - 1)d, \quad S_n = \frac{n}{2} \{2a + (n - 1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad r > 1, \quad S_n = \frac{a(1 - r^n)}{1 - r}, \quad r < 1, \quad S_\infty = \frac{a}{1 - r}, \quad |r| < 1$$

Binomial expansion:

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \dots + b^n, \text{ where } n \text{ is a positive integer.}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \times 2 \times \dots \times r} x^r + \dots \text{ where } n \text{ is a real number and } |x| < 1$$

Summations:

$$\sum_{r=1}^n r = \frac{1}{2} n(n+1), \quad \sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1), \quad \sum_{r=1}^n r^3 = \frac{1}{4} n^2(n+1)^2$$

Complex numbers:

$$z^n = (\cos x + i \sin x)^n = \cos nx + i \sin nx, \text{ where } n \text{ is an integer and } x \text{ is real}$$

$$e^{ix} = \cos x + i \sin x \text{ where } x \text{ is real}$$

$$[r(\cos x + i \sin x)]^n = r^n(\cos nx + i \sin nx)$$

Maclaurin's series:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{for all real } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r!} + \dots \quad (-1 < x \leq 1)$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^r}{r!} - \dots \quad (-1 \leq x < 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all real } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all real } x$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$$

Taylor's series:

$$f(x) = f(a) + f'(a) \frac{(x-a)}{1!} + f''(a) \frac{(x-a)^2}{2!} + f'''(a) \frac{(x-a)^3}{3!} + \dots + f^{(r)}(a) \frac{(x-a)^r}{r!} + \dots$$

$$\text{The trapezium rule } \int_a^b y dx = \frac{1}{2} h \{ (y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1}) \},$$

$$h = \frac{b-a}{n}, \text{ where } n \text{ is the number of intervals (strips)}$$

$$\text{The Newton-Raphson iteration } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

TRIGONOMETRY

$$\text{Sine Rule: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\text{Cosine rule } a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{Arc length of a circle: } s = r\theta, (\theta \text{ measured in radians})$$

Area of a sector of a circle: $\text{Area} = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$

If $\tan \frac{\alpha}{2} = t$, then $\sin \alpha = \frac{2t}{1+t^2}$ and $\cos \alpha = \frac{1-t^2}{1+t^2}$

Trigonometric Identities:

$$\cos^2 \alpha + \sin^2 \alpha \equiv 1, 1 + \tan^2 \alpha = \sec^2 \alpha, 1 + \cot^2 \alpha = \operatorname{cosec}^2 \alpha$$

$$\sin(\alpha \pm \beta) \equiv \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) \equiv \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) \equiv \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \quad \alpha \pm \beta \neq (k + \frac{1}{2})\pi$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\sin \alpha + \sin \beta \equiv 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha - \sin \beta \equiv 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta \equiv 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha - \cos \beta \equiv 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta - \alpha}{2} \quad \text{or} \quad -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

STATISTICS

Frequency distributions

$$\text{Mean } \bar{x} = \frac{\sum fx}{\sum f}$$

$$\text{Standard Deviation } \sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} = \sqrt{\sum_{i=1}^n \frac{(x_i - \bar{x})^2}{n}}$$

$$\text{Median } Q_2 = \left(\frac{n+1}{2}\right)^{\text{th}} \text{ value}$$

Grouped data

Mean ($\bar{x}$) = $\frac{\sum fx}{\sum f}$ where x = midpoint of each class, f is the frequency of each class.

Median = $l + \left(\frac{\frac{N}{2} - f_0}{f_1} \right) w$ where,

l = lower limit of the median class

N = total frequency

f_0 = frequency of class preceding the median class

f_1 = frequency of median class

w = width of median class

Mode = $l + \left(\frac{f_1 - f_2}{2f_1 - f_0 - f_2} \right) w$

Where ,

l = lower limit of the modal class

f_1 = frequency of the modal class

f_0 = frequency of the class preceding the modal class

f_2 = frequency of the class succeeding the modal class

w = width of the modal class

Measures of spread or dispersion

Standard deviation for the population

$$s.d = \sqrt{\frac{\sum fx^2 - \frac{(\sum fx)^2}{n}}{n}}$$

where x = midpoint of each class

f = the frequency of each class

n = population size.

unbiased estimator of the variance of X is $\hat{\sigma}^2 = \frac{n}{n-1} \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$

Product Moment Correlation Coefficient, r

$$r = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{\sqrt{\left[n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2 \right] \left[n \sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n y_i \right)^2 \right]}}$$

Covariance Formula $= \frac{S_{x_i y_i}}{S_{x_i} S_{y_i}}$ where $S_{x_i y_i}$ is the co-variance of x and y ,

$S_{x_i} S_{y_i}$ is the product of the standard deviation of x and y respectively

Regression line y on x

$y = a + bx$ passing through $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\sum x}{n} \text{ and } \bar{y} = \frac{\sum y}{n}$$

$$b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

$$b = \frac{S_{xy}}{S_{xx}}, \text{ where } S_{xx} \text{ is the variance of } x.$$

$$a = \bar{y} - b\bar{x}$$

MECHANICS

Uniformly accelerated motion

$$v = u + at, \quad s = \frac{1}{2}(u + v)t, \quad s = ut + \frac{1}{2}at^2, \quad v^2 = u^2 + 2as$$

Motion of a projectile

Equation of trajectory is:

$$y = x \tan \theta - \frac{gx^2}{2V^2 \cos^2 \theta}$$

$$= x \tan \theta - \frac{gx^2}{2V^2} (1 + \tan^2 \theta)$$

$$\text{Time of flight} = \frac{2V \sin \theta}{g}$$

$$\text{Greatest height} = \frac{V^2 \sin^2 \theta}{2g}$$

$$\text{Horizontal range} = \frac{V^2 \sin 2\theta}{g}, \text{ maximum range} = \frac{V^2}{g} \text{ for } \theta = \frac{\pi}{4}$$

Lami's Theorem

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}, \text{ where } F_1, F_2, F_3 \text{ are forces acting on a particle}$$

and α, β, γ are the angles vertically opposite F_1, F_2, F_3 , respectively



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